Appendix E

Martingales in continuous time

SECTION 1 explains the importance of sample path properties in the study of martingales, and other stochastic processes, in continuous time. Versions are defined. A delicate measurability question, regarding first hitting times on general Borel sets, is discussed. The notion of a standard filtration is introduced. The Section summarizes some definitions and results from the start of Chapter 6.

SECTION 2 presents the extension of the Stopping Time Lemma to submartingales with right continuous sample paths.

SECTION 3 shows how to construct a supermartingale with cadlag sample paths, by microsurgery on paths. Under a regularity condition on the filtration, each submartingale with right continuous expected value is shown to have a cadlag version.

SECTION 4 presents a remarkable property of the Brownian filtration.

1. Filtrations, sample paths, and stopping times

A ***stochastic process*** is a collection of random variables $\{X_t : t \in T\}$ all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The index set T is often referred to as "time." The theory of stochastic processes for "continuous time," where T is a subinterval of $\overline{\mathbb{R}}$, tends to be more complicated than for T countable (such as $T = \mathbb{N}$). The difficulties arise, in part, from problems related to management of uncountable families of negligible sets associated with uncountable collections of almost sure equality or inequality assertions. A nontrivial part of the continuous time theory deals with sample path properties—that is, with the behavior of a process $X_t(\omega)$ as a function of t for fixed ω—or with properties of X as a function of two variables, $X(t, \omega)$. Such properties are vital to many arguments based on approximation of processes through their values at a finite collections of times.

REMARK. I will treat the notations $X_t(\omega)$ and $X(t, \omega)$ as interchangeable. If ω is understood, I will also abbreviate to X_t or $X(t)$. The second form becomes more convenient when t is replaced by a more complicated expression: something like $X(t_0 + \tau'_{k-1})$ is much easier to read than $X_{t_0+\tau'_{k-1}(\omega)}(\omega)$.

Throughout this Appendix, T will usually denote $\mathbb{R}^+$ or a bounded interval, such as $[0, 1]$. The most desirable sample path properties are continuity, and a slightly weaker property that is known by the acronym for the French phrase meaning "continuous on the right with left limits."