Appendix F

Disintegration of measures

SECTION 1 decomposes a measure on a product space into a product of a marginal measure with a kernel.

SECTION 2 specializes the decomposition to the case of a measure concentrated on the graph of a function, establishing existence of a disintegration in the sense of Chapter 5.

1. Representation of measures on product spaces

Recall from Chapter 4 how we built a measure $\mu \otimes \Lambda$, out of a sigma-finite measure μ on $(\mathcal{X}, \mathcal{A})$ and a sigma-finite kernel $\Lambda := \{\lambda_t : t \in \mathcal{T}\}$, from $(\mathcal{T}, \mathcal{B})$ to $(\mathcal{X}, \mathcal{A})$, via an iterated integral,

$$(\mu \otimes \Lambda)\, f := \mu^t \lambda_t^x f(x,t) \qquad \text{for } f \text{ in } \mathcal{M}^+(\mathcal{X} \times \mathcal{T}, \mathcal{A} \otimes \mathcal{B}).$$

This Section treats the inverse problem: Given a measure μ on $\mathcal{B}$ and a measure Γ on $\mathcal{A} \otimes \mathcal{B}$, when does there exist a kernel Λ for which $\Gamma = \mu \otimes \Lambda$? Such representations are closely related to the problem of constructing conditional distributions, as you saw in Chapter 5.

<1> **Theorem.** *Let Γ be a sigma-finite measure on the product sigma-field $\mathcal{A} \otimes \mathcal{B}$ of a product space $\mathcal{X} \times \mathcal{T}$, and μ be a sigma-finite measure on $\mathcal{B}$. Suppose:*

(i) $\mathcal{X}$ is a metric space and $\mathcal{A}$ is its Borel sigma-field;

(ii) the $\mathcal{T}$-marginal of Γ is absolutely continuous with respect to μ;

(iii) $\Gamma = \sum_{i \in \mathbb{N}} \Gamma_i$, where each Γ_i is a finite measure concentrating on a set $\mathcal{X}_i \times \mathcal{T}$ with $\mathcal{X}_i$ compact.

Then there exists a kernel Λ from $(\mathcal{T}, \mathcal{B})$ to $(\mathcal{X}, \mathcal{A})$ for which $\Gamma = \mu \otimes \Lambda$. The kernel is unique up to a μ-equivalence.

REMARK. The uniqueness assertion means that, if $\widetilde{\Lambda} := \{\widetilde{\lambda}_t : t \in \mathcal{T}\}$ is another kernel for which $\Gamma = \mu \otimes \widetilde{\Lambda}$, then $\lambda_t = \widetilde{\lambda}_t$, as measures on $\mathcal{A}$, for μ almost all t.

Heuristics

Suppose for the moment that Γ has a representation as $\mu \otimes \Lambda$, for some kernel Λ. If we could characterize the kernel Λ in terms of Γ and μ alone, then we could try to construct Λ for a general Γ by reinterpreting the characterization as a definition.