Chapter 3

Densities and derivatives

SECTION 1 explains why the traditional split of introductory probability courses into two segments—the study of discrete distributions, and the study of "continuous" distributions—is unnecessary in a measure theoretic treatment. Absolute continuity of one measure with respect to another measure is defined. A simple case of the Radon-Nikodym theorem is proved.

*SECTION *2 establishes the Lebesgue decomposition of a measure into parts absolutely continuous and singular with respect to another measure, a result that includes the Radon-Nikodym theorem as a particular case.*

SECTION 3 shows how densities enter into the definitions of various distances between measures.

SECTION 4 explains the connection between the classical concept of absolute continuity and its measure theoretic generalization. Part of the Fundamental Theorem of Calculus is deduced from the Radon-Nikodym theorem.

*SECTION *5 establishes the Vitali covering lemma, the key to the identification of derivatives as densities.*

*SECTION *6 presents the proof of the other part of the Fundamental Theorem of Calculus, showing that absolutely continuous functions (on the real line) are Lebesgue integrals of their derivatives, which exist almost everywhere.*

1. Densities and absolute continuity

Nonnegative measurable functions create new measures from old.

Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a measure space, and let $\Delta(\cdot)$ be a function in $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$. The increasing, linear functional defined on $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$ by $\nu f := \mu(f\Delta)$ inherits from μ the Monotone Convergence property, which identifies it as an integral with respect to a measure on $\mathcal{A}$.

The measure μ is said to ***dominate*** ν; the measure ν is said to have ***density*** Δ with respect to μ. This relationship is often indicated symbolically as $\Delta = d\nu/d\mu$, which fits well with the traditional notation,

$$\int f(x)\, d\nu(x) = \int f(x) \frac{d\nu}{d\mu}\, d\mu(x).$$

The $d\mu$ symbols "cancel out," as in the change of variable formula for Lebesgue integrals.