Chapter 4

Product spaces and independence

SECTION 1 introduces independence as a property that justifies some sort of factorization of probabilities or expectations. A key factorization Theorem is stated, with proof deferred to the next Section, as motivation for the measure theoretic approach. The Theorem is illustrated by a derivation of a simple form of the strong law of large numbers, under an assumption of bounded fourth moments.

SECTION 2 formally defines independence as a property of sigma-fields. The key Theorem from Section 1 is used as motivation for the introduction of a few standard techniques for dealing with independence. Product sigma-fields are defined.

SECTION 3 describes a method for constructing measures on product spaces, starting from a family of kernels.

SECTION 4 specializes the results from Section 3 to define product measures. The Tonelli and Fubini theorems are deduced. Several important applications are presented.

*SECTION *5 discusses some difficulties encountered in extending the results of Sections 3 and 4 when the measures are not sigma-finite.*

SECTION 6 introduces a blocking technique to refine the proof of the strong law of large numbers from Section 1, to get a version that requires only a second moment condition.

*SECTION *7 introduces a truncation technique to further refine the proof of the strong law of large numbers, to get a version that requires only a first moment condition for identically distributed summands.*

*SECTION *8 discusses the construction of probability measures on products of countably many spaces.*

1. Independence

Much classical probability theory, such as the laws of large numbers and central limit theorems, rests on assumptions of independence, which justify factorizations for probabilities of intersections of events or expectations for products of random variables.

An elementary treatment usually starts from the definition of independence for events. Two events A and B are said to be independent if $\mathbb{P}(AB) = (\mathbb{P}A)(\mathbb{P}B)$; three events A, B, and C, are said to be independent if not only $\mathbb{P}(ABC) = (\mathbb{P}A)(\mathbb{P}B)(\mathbb{P}C)$ but also $\mathbb{P}(AB) = (\mathbb{P}A)(\mathbb{P}B)$ and $\mathbb{P}(AC) = (\mathbb{P}A)(\mathbb{P}C)$ and $\mathbb{P}(BC) = (\mathbb{P}B)(\mathbb{P}C)$. And so on. There are similar definitions for independence of random variables, in terms of joint distribution functions or joint densities. The definitions have two