Solutions to sheet 4

(4.1) An appropriate sample space consists of four outcomes:

$$
\begin{aligned}
s_1 &= \text{prize in box A, host reveals B}\\
s_2 &= \text{prize in box A, host reveals C}\\
s_3 &= \text{prize in box B, host reveals C}\\
s_4 &= \text{prize in box C, host reveals B}
\end{aligned}
$$

The assumption about how the prize was located corresponds to the probability assignment

$$\mathbb{P}\{s_1\} + \mathbb{P}\{s_2\} = \mathbb{P}\{s_3\} = \mathbb{P}\{s_4\} = \frac{1}{3}$$

There is nothing in the problem that shows how to split the 1/3 between s_1 and s_2, so I will assume that when the host has a choice he mentally flips a fair coin. That assumption gives

$$(*) \qquad \mathbb{P}\{s_1\} = \mathbb{P}\{s_2\} = \frac{1}{6}$$

The events of interest are

$$\mathcal{A} = \{\text{ prize in box A }\} = \{s_1\} \text{ OR } \{s_2\}$$

and

$$\mathcal{B} = \{\text{ host reveals box B }\} = \{s_1\} \text{ OR } \{s_4\}$$

From the rules for probabilities,

$$\mathbb{P}(\mathcal{A} \mid \mathcal{B}) = \frac{\mathbb{P}(\mathcal{A} \text{ AND } \mathcal{B})}{\mathbb{P}(\mathcal{B})} = \frac{\mathbb{P}\{s_1\}}{\mathbb{P}\{s_1\} + \mathbb{P}\{s_4\}} = \frac{1}{3}$$

Fred switches to box C because it has a higher conditional probability of containing the prize. Do you see how Fred gained information from the fact that the host did not choose box C?

You might wonder what would happen if you replaced (*) by another assumption. For example, suppose the host had decided to choose B if it was possible to do so without revealing the location of the prize. That assumption would give $\mathbb{P}\{s_1\} = 1/3$ and $\mathbb{P}\{s_2\} = 0$, from which it would follow that $\mathbb{P}(\mathcal{A} \mid \mathcal{B}) = 1/2$. If Fred were thinking that way, it would be whimsical of him to switch.

(4.2) For this problem a sample space is not really necessary: you could just apply the rules of probability to the events $F = \{$ choose fair coin$\}$, $B = \{$ choose heavy headed coin $\}$, H_1, and H_2. Interpret "choose a coin at random" to mean $\mathbb{P}(F) = \mathbb{P}(B) = 1/2$. Given the choice of coin, the two tosses are conditionally independent,

$$
\begin{aligned}
\mathbb{P}(H_1 \text{ AND } H_2 \mid F) &= \frac{1}{4}\\
\mathbb{P}(H_1 \text{ AND } H_2 \mid B) &= \frac{4}{9}
\end{aligned}
$$

Split the event H_1 into two disjoint pieces, then condition on the choice of coin to get

$$
\begin{aligned}
\mathbb{P}(H_1) &= \mathbb{P}(H_1 \text{ AND } F) + \mathbb{P}(H_1 \text{ AND } B)\\
&= \mathbb{P}(H_1 \mid F)\mathbb{P}(F) + \mathbb{P}(H_1 \mid B)\mathbb{P}(B)\\
&= \frac{1}{2} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} = \frac{7}{12}
\end{aligned}
$$

A similar calculation gives $\mathbb{P}(H_2) = 7/12$.

Split the event H_1 AND H_2 according to the choice of coin to get

$$
\begin{aligned}
\mathbb{P}(H_1 \text{ AND } H_2) &= \mathbb{P}(H_1 \text{ AND } H_2 \mid F)\mathbb{P}(F) + \mathbb{P}(H_1 \text{ AND } H_2 \mid B)\mathbb{P}(B)\\
&= \frac{1}{4} \times \frac{1}{2} + \frac{4}{9} \times \frac{1}{2} = \frac{25}{72}
\end{aligned}
$$

Run the conditioning rule backwards to deduce that

$$\mathbb{P}(H_2 \mid H_1) = \frac{\mathbb{P}(H_1 \text{ AND } H_2)}{\mathbb{P}(H_1)} = \frac{25}{42} \neq \mathbb{P}(H_2)$$

The events H_1 and H_2 are not independent.

You might find it easier to understand this result if you determine what you learn about the coin from the result of the first toss:

$$\mathbb{P}(F \mid H_1) = \frac{\mathbb{P}(H_1 \mid F)\mathbb{P}(F)}{\mathbb{P}(H_1 \mid F)\mathbb{P}(F) + \mathbb{P}(H_1 \mid B)\mathbb{P}(B)} = \frac{3}{7}$$

Then calculate

$$\mathbb{P}(H_2 \mid H_1) = \mathbb{P}(H_2 \mid H_1 \text{ AND } F)\mathbb{P}(F \mid H_1) + \mathbb{P}(H_2 \mid H_1 \text{ AND } B)\mathbb{P}(B \mid H_1)$$

(Can you derive this relationship?) Given the choice of coin, the events H_1 and H_2 are conditionally independent, so that $\mathbb{P}(H_2 \mid H_1 \text{ AND } F) = \mathbb{P}(H_2 \mid F) = 1/2$, and similarly for $\mathbb{P}(H_2 \mid H_1 \text{ AND } B)$. The last expression then simplifies to

$$\frac{1}{2} \times \frac{3}{7} + \frac{2}{3} \times \frac{4}{7} = \frac{25}{42},$$

as before. Don't worry about this alternative, longer form of the argument if you are not comfortable with conditioning.

You might also gain some understanding of why the result of the first toss gives some useful information about the result of the second toss by considering a more extreme example. Suppose the coins had probabilities 0.001 and 0.999 of landing heads. If the first toss with the coin gave a head, what would you predict the second toss to give?

If you really feel the need for a sample space, you could choose one with eight points, fh_1h_2, fh_1t_2, ..., bt_1t_2, where h_i and t_i denote heads or tails on the ith toss, and f and b denote the choice of coin. With this sample space,

$$H_1 = \{fh_1h_2\} \text{ OR } \{fh_1t_2\} \text{ OR } \{bh_1h_2\} \text{ OR } \{bh_1t_2\},$$

and so on. In fact each outcome is determined by the intersection of three events,

$$fh_1h_2 = F \text{ AND } H_1 \text{ AND } H_2,$$

and so on. The assignment of probabilities to outcomes follows from the conditioning rule,

$$\mathbb{P}\{fh_1h_2\} = \mathbb{P}(F)\mathbb{P}(H_1 \mid F)\mathbb{P}(H_2 \mid F \text{ AND } H_2) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

and so on:

$$\mathbb{P}\{fh_1h_2\} = \mathbb{P}\{fh_1t_2\} = \mathbb{P}\{fh_1t_2\} = \mathbb{P}\{ft_1t_2\} = \frac{1}{8}$$

$$\mathbb{P}\{bh_1h_2\} = \frac{2}{9}, \quad \mathbb{P}\{bh_1t_2\} = \mathbb{P}\{bt_1h_2\} = \frac{1}{9}, \quad \mathbb{P}\{bt_1t_2\} = \frac{1}{18}$$

The calculations are much the same as before.

(4.3) A suitable sample space to describe a single family would be

outcome	m	fm	ffm	fffm	ffff
probability	1/2	1/4	1/8	1/16	1/16
value of X	0	1	2	3	4
value of Y	1	1	1	1	0

Here f denotes birth of a female, and m denotes birth of a male, with ordering as shown. [Many of you wanted to use X and Y instead of f and m. With such a scheme, X would have two different meanings: both the number of females and the occurence of a female birth. It is easy to get confused when the same symbol can have different meanings.] The two added rows show the values taken by X and Y at each outcome. Notice that the event $\{Y = 1\}$ is made up of four outcomes.

Distribution of X:

value of X	0	1	2	3	4
probability	1/2	1/4	1/8	1/16	1/16

Distribution of Y:

value of Y	0	1
probability	1/16	15/16

From the two tables showing the distributions, we get

$$\mu_X = 0 \times \frac{1}{2} + 1 \times \frac{1}{4} + \ldots + 4 \times \frac{1}{6} = \frac{15}{16}$$
$$\mu_Y = 0 \times \frac{1}{16} + 1 \times \frac{15}{16} = \frac{15}{16}$$

Surprised?

The two probabilities for part (iv) come directly from the first table:

$$\mathbb{P}\{X > Y\} = \mathbb{P}\{ffm\} + \mathbb{P}\{fffm\} + \mathbb{P}\{ffff\} = \frac{1}{4}$$
$$\mathbb{P}\{X < Y\} = \mathbb{P}\{m\} = \frac{1}{2}$$

Just as a check, note that $\mathbb{P}\{X = Y\} = \mathbb{P}\{fm\} = 1/4$.

The means give the relevant figures for judging the new policy. Taken over a large number of families, which are assumed to reproduce independently of each other, the average number of males per family and the average number of females per family will both be close to 15/16. The new policy should have the effect of bringing the proportions of males and females in the whole population closer to each other.

The wording of the question is not clear regarding the meaning of success. Is it a success if the proportions of males and females move closer to each other? If so, how is it that the surplus of males arose in the first place? If success means that the proportion of males should be brought below the proportion of females, then the policy will not succeed.

Why is it that the answers to part (iv) do not provide the answer to part (v)? If you knew that about half the families had more males than females, and only a quarter of the families had more females than males, why wouldn't it follow that males would be in the majority? [Is it relevant to ask about how many more females than males, or males than females?]