Chapter 9

Poisson processes

The Binomial distribution and the geometric distribution describe the behavior of two random variables derived from the random mechanism that I have called coin tossing. The name *coin tossing* describes the whole mechanism; the names *Binomial* and *geometric* refer to particular aspects of that mechanism. If we increase the tossing rate to n tosses per second and decrease the probability of heads to a small p, while keeping the expected number of heads per second fixed at $\lambda = np$, the number of heads in a t second interval will have approximately a Bin(nt, p) distribution, which is close to the Poisson(λt). Also, the numbers of heads tossed during disjoint time intervals will still be independent random variables. In the limit, as $n \to \infty$, we get an idealization called a **Poisson process**.

REMARK. The double use of the name Poisson is unfortunate. Much confusion would be avoided if we all agreed to refer to the mechanism as "idealized-very-fast-coin-tossing", or some such. Then the Poisson distribution would have the same relationship to idealized-very-fast-coin-tossing as the Binomial distribution has to coin-tossing. Obversely, we could create more confusion by renaming coin tossing as "the binomial process". Neither suggestion is likely to be adopted, so you should just get used to having two closely related objects with the name Poisson.

Definition. *A Poisson process with rate λ on $[0, \infty)$ is a random mechanism that generates "points" strung out along $[0, \infty)$ in such a way that*

(i) the number of points landing in any subinterval of length t is a random variable with a Poisson(λt) distribution

(ii) the numbers of points landing in disjoint (= non-overlapping) intervals are independent random variables.

Note that, for a very short interval of length δ, the number of points X in the interval has a Poisson$(\lambda\delta)$ distribution, with

$$\begin{aligned}
\mathbb{P}\{X = 0\} &= e^{-\lambda\delta} = 1 - \lambda\delta + \text{ terms of order } \delta^2 \text{ or smaller} \\
\mathbb{P}\{X = 1\} &= \lambda\delta e^{-\lambda\delta} = \lambda\delta + \text{ terms of order } \delta^2 \text{ or smaller} \\
\mathbb{P}\{X \geq 2\} &= 1 - e^{-\lambda\delta} - \lambda\delta e^{-\lambda\delta} = \text{ quantity of order } \delta^2.
\end{aligned}$$

When we pass to the idealized mechanism of points generated in continuous time, several awkward artifacts of discrete-time coin tossing disappear.

Example <9.1>: (Gamma distribution from Poisson process) The waiting time W_k to the kth point in a Poisson process with rate λ has a continuous distribution, with density $g_k(w) = \lambda^k w^{k-1} e^{-\lambda w}/(k-1)!$ for $w > 0$, zero otherwise.

It is easier to remember the distribution if we rescale the process, defining $T_k = \lambda W_k$. From the results derived in Homework Problem 6.4, the new T_k has a continuous distribution with a **gamma(k) density**,

$$f_k(t) = \frac{t^{k-1}e^{-t}}{(k-1)!}\mathbb{I}\{t > 0\}$$

REMARK. Notice that $g_k = f_k$ when $\lambda = 1$. That is, T_k is the waiting time to the kth point for a Poisson process with rate 1. Put another way, we can generate a Poisson process with rate λ by taking the points appearing at times $0 < T_1 < T_2 < T_3 < \ldots$ from a Poisson process with rate 1, then rescaling to produce a new process with points at

$$0 < \frac{T_1}{\lambda} < \frac{T_2}{\lambda} < \frac{T_3}{\lambda} < \ldots$$

You could verify this assertion by checking the two defining properties for a Poisson process with rate λ. Doesn't it makes sense that, as λ gets bigger, the points appear more rapidly?

More generally, for each $\alpha > 0$,

$$f_\alpha(t) = \begin{cases} \dfrac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)} & \text{for } t > 0, \\ 0 & \text{otherwise,} \end{cases}$$

is called the **gamma(α) density**. The scaling constant, $\Gamma(\alpha)$, which ensures that the density integrates to one, is given by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx \qquad \text{for each } \alpha > 0.$$

The function $\Gamma(\cdot)$ is called the **gamma function**. Don't confuse the gamma density (or the gamma distribution that it defines) with the gamma function.

Example <9.2>: Facts about the gamma function: $\Gamma(k) = (k-1)!$ for $k = 1, 2, \ldots$, and $\Gamma(1/2) = \sqrt{\pi}$.

The change of variable used in Example <9.2> to prove $\Gamma(1/2) = \sqrt{\pi}$ is essentially the same piece of mathematics as the calculation used to find the density for the distribution of $Y = Z^2/2$ when $Z \sim N(0,1)$: For $y > 0$, and small $\delta > 0$,

$$\begin{aligned} \mathbb{P}\{y < Y < y+\delta\} &= \mathbb{P}\{2y < Z^2 < 2y+2\delta\} \\ &= \mathbb{P}\{\sqrt{2y} < Z < \sqrt{2y+2\delta}\} + \mathbb{P}\{-\sqrt{2y+2\delta} < Z < -\sqrt{2y}\} \\ &\approx \frac{2\delta}{\sqrt{2y}}\phi(\sqrt{2y}) \qquad \text{because } \sqrt{2y+2\delta} - \sqrt{2y} \approx \delta/\sqrt{2y} \\ &= \delta\frac{1}{\sqrt{\pi}}y^{-1/2}e^{-y} \end{aligned}$$

That is, Y has a gamma (1/2) distribution. □

Example <9.3>: Moments of the gamma distribution

The special case of the gamma distribution when the parameter α equals 1 is called the **(standard) exponential distribution**, with density $f_1(t) = e^{-t}$ for $t > 0$, and zero elsewhere. From Example <9.3>, if T_1 has a standard exponential distribution then $\mathbb{E}T_1 = 1$. The waiting time W_1 to the first point in a Poisson process with rate λ has the same distribution as T_1/λ, that is, a continuous distribution with density $\lambda e^{-\lambda t}$ for $t > 0$, an **exponential distribution with expected value $1/\lambda$**. Don't confuse the exponential density (or the exponential distribution that it defines) with the exponential function.

Notice the parallels between the negative binomial distribution (in discrete time) and the gamma distribution (in continuous time). Each distribution corresponds to the waiting time to the kth occurrence of something, for various values of k. The negative binomial (see Problem Sheet 4) can be written as a sum of independent random variables, each with a geometric distribution. The gamma(k) can be written as a sum of k independent random variables,

$$T_k = T_1 + (T_2 - T_1) + (T_3 - T_2) + \ldots + (T_k - T_{k-1}),$$

each with a standard exponential distribution. (For a Poisson process, the independence between the counts in disjoint intervals ensures that the mechanism determining the time $W_2 - W_1$ between the first and the second points is just another Poisson process started off at time W_1. And so on.) The times between points in a Poisson process are independent, exponentially distributed random variables.

Poisson Processes can also be defined for sets other than the half-line.

Example <9.4>: A Poisson Process in two dimensions.

Things to remember

- Analogies between coin tossing, as a discrete time mechanism, and the Poisson process, as a continuous time mechanism:

DISCRETE TIME	CONTINUOUS TIME
coin tossing, prob p of heads	Poisson process with rate λ
$X = \#\text{heads in } n \text{ tosses} \sim \text{Bin}(n,p)$ $\mathbb{P}\{X=i\} = \binom{n}{i} p^i q^{n-i}$ for $i = 0,1,\ldots,n$	$X = \#\text{ points in } [a, a+t] \sim \text{Poisson}(\lambda t)$ $\mathbb{P}\{X=i\} = e^{-\lambda t}(\lambda t)^i/i!$ for $i = 0,1,2\ldots$
(geometric) $N_1 = \#$ tosses to first head; $\mathbb{P}\{N_1 = 1+i\} = q^i p$ for $i = 0,1,2,\ldots$	(exponential) $T_1/\lambda =$ time to first point; T_1 has density $f_1(t) = e^{-t}$ for $t > 0$
(negative binomial) $N_k = \#$ tosses to kth head; $\mathbb{P}\{N_k = k+i\} = \binom{k+i-1}{k-1} q^i p^k = \binom{-k}{i}(-q)^i p^k$ for $i = 0,1,2,\ldots$	(gamma) $T_k/\lambda =$ time to kth point; T_k has density $f_k(t) = t^{k-1}e^{-t}/k!$ for $t > 0$
negative binomial as sum of independent geometrics	gamma(k) as sum of independent exponentials

Examples for Chapter 9

<9.1> **Example.** Let W_k denote the waiting time to the kth point in a Poisson process on $[0,\infty)$ with rate λ. It has a continuous distribution, whose density g_k we can find by an argument similar to the one used in Chapter 6 to find the distribution of an order statistic for a sample from the Uniform(0, 1).

For a given $w > 0$ and small $\delta > 0$, write M for the number of points landing in the interval $[0,w)$, and N for the number of points landing in the interval $[w, w+\delta]$. From the definition of a Poisson process, M and N are independent random variables with

$$M \sim \text{Poisson}(\lambda w) \qquad \text{and} \qquad N \sim \text{Poisson}(\lambda\delta).$$

To have W_k lie in the interval $[w, w+\delta]$ we must have $N \geq 1$. When $N = 1$, we need exactly $k-1$ points to land in $[0, w)$. Thus

$$\mathbb{P}\{w \leq W_k \leq w+\delta\} = \mathbb{P}\{M = k-1,\ N = 1\} + \mathbb{P}\{w \leq W_k \leq w+\delta,\ N \geq 2\}.$$

The second term on the right-hand side is of order δ^2. Independence of M and N lets us factorize the contribution from $N = 1$ into

$$\begin{aligned}\mathbb{P}\{M = k-1\}\mathbb{P}\{N = 1\} &= \frac{e^{-\lambda w}(\lambda w)^{k-1}}{(k-1)!}\frac{e^{-\lambda\delta}(\lambda\delta)^1}{1!} \\ &= \frac{e^{-\lambda w}\lambda^{k-1}w^{k-1}}{(k-1)!}\left(\lambda\delta + \text{ smaller order terms}\right),\end{aligned}$$

Thus

$$\mathbb{P}\{w \leq W_k \leq w+\delta\} = \frac{e^{-\lambda w}\lambda^k w^{k-1}}{(k-1)!}\delta + \text{ smaller order terms},$$

which makes

$$g_k(w) = \frac{e^{-\lambda w}\lambda^k w^{k-1}}{(k-1)!} \qquad \text{for } w > 0.$$

the density function for W_k. □

<9.2> **Example.** The gamma function is defined for $\alpha > 0$ by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx.$$

By direct integration, $\Gamma(1) = \int_0^\infty e^{-x}dx = 1$. Also, a change of variable $y = \sqrt{2x}$ gives

$$\begin{aligned}\Gamma(1/2) &= \int_0^\infty x^{-1/2}e^{-x}\,dx \\ &= \int_0^\infty \sqrt{2}e^{-y^2/2}\,dy \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{2\pi}}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-y^2/2}\,dy \\ &= \sqrt{\pi} \qquad \text{cf. integral of} N(0,1) \text{ density.}\end{aligned}$$

For each $\alpha > 0$, an integration by parts gives

$$\begin{aligned}\Gamma(\alpha+1) &= \int_0^\infty x^\alpha e^{-x}\,dx \\ &= \left[-x^\alpha e^{-x}\right]_0^\infty + \alpha\int_0^\infty x^{\alpha-1}e^{-x}\,dx \\ &= \alpha\Gamma(\alpha).\end{aligned}$$

Repeated appeals to the same formula, for $\alpha > 0$ and each positive integer m, give

$$(*) \qquad \Gamma(\alpha+m) = (\alpha+m-1)(\alpha+m-2)\ldots(\alpha)\Gamma(\alpha).$$

In particular,

$$\Gamma(k) = (k-1)(k-2)(k-3)\ldots(2)(1)\Gamma(1) = (k-1)! \qquad \text{for } k = 1, 2, \ldots.$$

□

<9.3> **Example.** For parameter value $\alpha > 0$, the gamma(α) distribution is defined by its density

$$f_\alpha(t) = \begin{cases} t^{\alpha-1}e^{-t}/\Gamma(\alpha) & \text{for } t > 0 \\ 0 & \text{otherwise}\end{cases}$$

If a random variable T has a gamma(α) distribution then, for each positive integer m,

$$\begin{aligned}\mathbb{E}T^m &= \int_0^\infty t^m f_\alpha(t)\,dt \\ &= \int_0^\infty \frac{t^m t^{\alpha-1}e^{-t}}{\Gamma(\alpha)}\,dt \\ &= \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)} \\ &= (\alpha+m-1)(\alpha+m-2)\dots(\alpha) \qquad \text{by equality } (*) \text{ in Example <9.2>.}\end{aligned}$$

In particular, $\mathbb{E}T = \alpha$ and

$$\mathrm{var}(T) = \mathbb{E}\left(T^2\right) - (\mathbb{E}T)^2 = (\alpha+1)\alpha - \alpha^2 = \alpha.$$

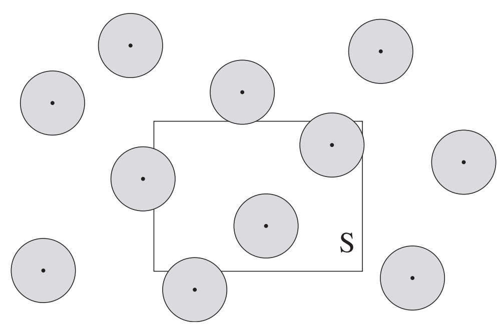
□

<9.4> **Example.** A Poisson process with rate λ on $\mathbb{R}^2$ is a random mechanism that generates "points" in the plane in such a way that

(i) the number of points landing in any region of area A is a random variable with a Poisson(λA) distribution

(ii) the numbers of points landing in disjoint regions are independent random variables.

Suppose mold spores are distributed across the plane as a Poisson process with intensity λ. Around each spore, a circular moldy patch of radius r forms. Let S be some bounded region. Find the expected proportion of the area of S that is covered by mold.



Write $\mathbf{x} = (x, y)$ for the typical point of $\mathbb{R}^2$. If B is a subset of $\mathbb{R}^2$,

$$\text{area of } S \cap B = \iint_{\mathbf{x}\in S} \mathbb{I}\{\mathbf{x} \in B\}\,d\mathbf{x}$$

If B is a random set then

$$\mathbb{E}\left(\text{ area of } S \cap B\right) = \iint_{\mathbf{x}\in S} \mathbb{E}\mathbb{I}\{\mathbf{x} \in B\}\,d\mathbf{x} = \iint_{\mathbf{x}\in S} \mathbb{P}\{\mathbf{x} \in B\}\,d\mathbf{x}$$

If B denotes the moldy region of the plane,

$$\begin{aligned}1 - \mathbb{P}\{\mathbf{x} \in B\} &= \mathbb{P}\{\text{ no spores land within a distance } r \text{ of } \mathbf{x}\ \} \\ &= \mathbb{P}\{\text{ no spores in circle of radius } r \text{ around } \mathbf{x}\ \} \\ &= \exp\left(-\lambda\pi r^2\right)\end{aligned}$$

Notice that the probability does not depend on $\mathbf{x}$. Connsequently,

$$\mathbb{E}\left(\text{ area of } S \cap B\right) = \iint_{\mathbf{x}\in S} 1 - \exp\left(-\lambda\pi r^2\right)\,d\mathbf{x} = \left(1 - \exp\left(-\lambda\pi r^2\right)\right) \times \text{ area of } S$$

The expected proportion of the area of S that is covered by mold is $1 - \exp\left(-\lambda\pi r^2\right)$. □

