Chapter 12

Conditional densities

12.1 Overview

Density functions determine continuous distributions. If a continuous distribution is calculated conditionally on some information, then the density is called a ***conditional density***. When the conditioning information involves another random variable with a continuous distribution, the conditional density can be calculated from the joint density for the two random variables.

Suppose X and Y have a jointly continuous distribution with joint density $f(x, y)$. From Chapter 11, you know that the marginal distribution of X is continuous with density

$$g(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

The conditional distribution for Y given $X = x$ has a (conditional) density, which I will denote by $h(y \mid X = x)$, or just $h(y \mid x)$ if the conditioning variable is unambiguous, for which

$$\mathbb{P}\{y \le Y \le y + \delta \mid X = x\} \approx \delta h(y \mid X = x), \qquad \text{for small } \delta > 0.$$

Conditioning on $X = x$ should be almost the same as conditioning on the event $\{x \le X \le x + \epsilon\}$ for a very small $\epsilon > 0$. That is, provided $g(x) > 0$,

$$\begin{aligned} \mathbb{P}\{y \le Y \le y + \delta \mid X = x\} &\approx \mathbb{P}\{y \le Y \le y + \delta \mid x \le X \le x + \epsilon\} \\ &= \frac{\mathbb{P}\{y \le Y \le y + \delta, x \le X \le x + \epsilon\}}{\mathbb{P}\{x \le X \le x + \epsilon\}} \\ &\approx \frac{\delta \epsilon f(x, y)}{\epsilon g(x)}. \end{aligned}$$

In the limit, as ϵ tends to zero, we are left with $\delta \approx \delta f(x,y)/g(x)$. That is,

$$h(y \mid X = x) = f(x,y)/g(x) \qquad \text{for each } x \text{ with } g(x) > 0.$$

Less formally, the conditional density is

$$\boxed{h(y \mid X = x) = \frac{\text{joint } (X,Y) \text{ density at } (x,y)}{\text{marginal } X \text{ density at } x}}$$

The first Example illustrates two ways to find a conditional density: first by calculation of a joint density followed by an appeal to the formula for the conditional density; and then by a sneakier method where all the random variables are built directly using polar coordinates.

Example <12.1> Let X and Y be independent random variables, each distributed $N(0,1)$. Define $R = \sqrt{X^2+Y^2}$. Show that, for each $r > 0$, the conditional distribution of X given $R = r$ has density

$$h(x \mid R = r) = \frac{\mathbf{1}\{|x| < r\}}{\pi\sqrt{r^2 - x^2}} \qquad \text{for } r > 0.$$

The most famous example of a continuous condition distribution comes from pairs of random variables that have a bivariate normal distribution. For each constant $\rho \in (-1,+1)$, the ***standard bivariate normal with correlation*** ρ is defined as the joint distribution of a pair of random variables constructed from independent random variables X and Y, each distributed $N(0,1)$. Define $Z = \rho X + \sqrt{1-\rho^2}\,Y$. The pair X, Y has a jointly continuous distribution with density $f(x,y) = (2\pi)^{-1}\exp\big(-(x^2+y^2)/2\big)$. Apply the result from Example <11.4> with

$$(X,Z) = (X,Y)A \qquad \text{where } A = \begin{pmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{pmatrix}$$

to deduce that X, Z have joint density

$$f_\rho(x,z) = \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xz + z^2}{1-\rho^2}\right).$$

Notice the symmetry in x and z. The X and Z marginals must be the same. Thus $Z \sim N(0,1)$. Also

$$\begin{aligned} \operatorname{cov}(X,Z) &= \operatorname{cov}(X, \rho X + \sqrt{1-\rho^2}\,Y) \\ &= \rho \operatorname{cov}(X,X) + \sqrt{1-\rho^2}\operatorname{cov}(X,Y) = \rho. \end{aligned}$$

Remark. The ***correlation*** between two random variables S and T is defined as

$$\operatorname{corr}(S,T) = \frac{\operatorname{cov}(S,T)}{\sqrt{\operatorname{var}(S)\operatorname{var}(T)}}.$$

If $\operatorname{var}(S) = \operatorname{var}(T) = 1$ the correlation reduces to the covariance.

By construction, the conditional distribution of Z given $X = x$ is just the conditional distribution of $\rho x + \sqrt{1-\rho^2}Y$ given $X = x$. Independence of X and Y then shows that

$$Z \mid X = x \sim N(\rho x, 1-\rho^2).$$

In particular, $\mathbb{E}(Z \mid X = x) = \rho x$. By symmetry of f_ρ, we also have $X \mid Z = z \sim N(\rho z, 1-\rho^2)$, a fact that you could check by dividing $f_\rho(x,z)$ by the standard normal density for Z.

Example <12.2> Let S denote the height (in inches) of a randomly chosen father, and T denote the height (in inches) of his son at maturity. Suppose each of S and T has a $N(\mu, \sigma^2)$ distribution with $\mu = 69$ and $\sigma = 2$. Suppose also that the standardized variables $(S-\mu)/\sigma$ and $(T-\mu)/\sigma$ have a standard bivariate normal distribution with correlation $\rho = .3$.

If Sam has a height of $S = 74$ inches, what would one predict about the ultimate height T of his young son Tom?

For the standard bivariate normal, if the variables are uncorrelated (that is, if $\rho = 0$) then the joint density factorizes into the product of two $N(0,1)$ densities, which implies that the variables are independent. This situation is one of the few where a zero covariance (zero correlation) implies independence.

The final Example demonstrates yet another connection between Poisson processes and order statistics from a uniform distribution. The arguments make use of the obvious generalizations of joint densities and conditional densities to more than two dimensions.

Definition. Say that random variables X, Y, Z have a jointly continuous distribution with joint density $f(x,y,z)$ if

$$\mathbb{P}\{(X,Y,Z) \in A\} = \iiint_A f(x,y,z)\,dx\,dy\,dz \qquad \text{for each } A \subseteq \mathbb{R}^3.$$

As in one and two dimensions, joint densities are typically calculated by looking at small regions: for a small region Δ around (x_0, y_0, z_0)

$$\mathbb{P}\{(X,Y,Z) \in \Delta\} \approx (\text{volume of } \Delta) \times f(x_0, y_0, z_0).$$

Similarly, the joint density for (X,Y) conditional on $Z = z$ is defined as the function $h(x, y \mid Z = z)$ for which

$$\mathbb{P}\{(X,Y) \in B \mid Z = z\} = \iiint \mathbb{I}\{(x,y) \in B\} h(x, y \mid Z = z)\, dx\, dy$$

for each subset B of $\mathbb{R}^2$. It can be calculated, at z values where the marginal density for Z,

$$g(z) = \iint_{\mathbb{R}^2} f(x,y,z)\, dx\, dy,$$

is strictly positive, by yet another small-region calculation. If Δ is a small subset containing (x_0, y_0) then, for small $\epsilon > 0$,

$$\begin{aligned}
\mathbb{P}\{(X,Y) \in \Delta \mid Z = z_0\} &\approx \mathbb{P}\{(X,Y) \in \Delta \mid z_0 \le Z \le z_0 + \epsilon\} \\
&= \frac{\mathbb{P}\{(X,Y) \in \Delta,\, z_0 \le Z \le z_0 + \epsilon\}}{\mathbb{P}\{z_0 \le Z \le z_0 + \epsilon\}} \\
&\approx \frac{((\text{area of } \Delta) \ \times \epsilon) f(x_0, y_0, z_0)}{\epsilon g(z_0)} \\
&= (\text{area of } \Delta)\ \frac{f(x_0, y_0, z_0)}{g(z_0)}.
\end{aligned}$$

Remark. Notice the identification of the set of points (x,y,z) in $\mathbb{R}^3$ for which $(x,y) \in \Delta$ and $z_0 \le z \le z_0 + \epsilon$ as a small region with volume equal to (area of Δ) $\times\, \epsilon$.

That is, the conditional (joint) distribution of (X,Y) given $Z = z$ has density

$$h(x, y \mid Z = z) = \frac{f(x,y,z)}{g(z)} \qquad \text{provided } g(z) > 0.$$

Remark. Many authors (including me) like to abbreviate $h(x, y \mid Z = z)$ to $h(x, y \mid z)$. Many others run out of symbols and write $f(x, y \mid z)$ for the conditional (joint) density of (X,Y) given $Z = z$. This notation is defensible if one can somehow tell which values are being conditioned on. In a problem with lots of conditioning it can get confusing to remember which f is the joint density and which is conditional on something. To avoid confusion, some authors write things like $f_{X,Y|Z}(x, y \mid z)$ for the conditional density and $f_X(x)$ for the X-marginal density, at the cost of more cumbersome notation.

Example <12.3> Let T_i denote the time to the ith point in a Poisson process with rate λ on $[0, \infty)$. Find the joint distribution of (T_1, T_2) conditional on T_3.

From the result in the previous Example, you should be able to deduce that, conditional on $T_3 = t_3$ for a given $t_3 > 0$, the random variables $(T_1/T_3, T_2/T_3)$ are uniformly distributed over the triangular region

$$\{(u_1 u_2) \in \mathbb{R}^2 : 0 < u_1 < u_2 < 1\}.$$

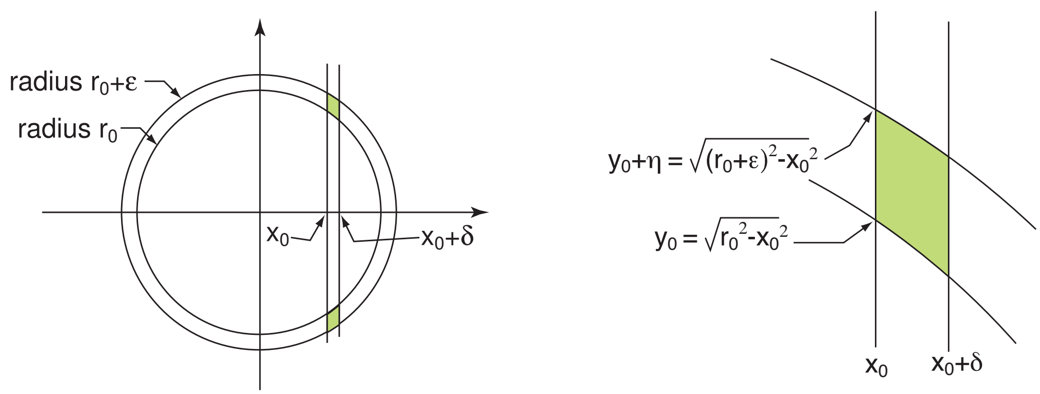
HW11 will step you through an analogous result for order statistics.

12.2 Examples for Chapter 12

<12.1> **Example.** *Let X and Y be independent random variables, each distributed $N(0,1)$. Define $R = \sqrt{X^2 + Y^2}$. For each $r > 0$, find the density for the conditional distribution of X given $R = r$.*

The joint density for (X, Y) equals $f(x, y) = (2\pi)^{-1} \exp\left(-(x^2 + y^2)/2\right)$. To find the conditional density for X given $R = r$, first I'll find the joint density ψ for X and R, then I'll calculate its X marginal, and then I'll divide to get the conditional density. A simpler method is described at the end of the Example.

We need to calculate $\mathbb{P}\{x_0 \le X \le x_0 + \delta,\, r_0 \le R \le r_0 + \epsilon\}$ for small, positive δ and ϵ. For $|x_0| < r_0$, the event corresponds to the two small regions in the (X, Y)-plane lying between the lines $x = x_0$ and $x = x_0 + \delta$, and between the circles centered at the origin with radii r_0 and $r_0 + \epsilon$.



By symmetry, both regions contribute the same probability. Consider the upper region. For small δ and ϵ, the region is approximately a parallelogram, with side length $\eta = \sqrt{(r_0+\epsilon)^2 - x_0^2} - \sqrt{r_0^2 - x_0^2}$ and width δ. We could expand the expression for η as a power series in ϵ by multiple applications of Taylor's theorem. It is easier to argue less directly, starting from the equalities

$$x_0^2 + (y_0+\eta)^2 = (r_0+\epsilon)^2 \qquad \text{AND} \qquad x_0^2 + y_0^2 = r_0^2.$$

Take differences to deduce that $2y_0\eta + \eta^2 = 2r_0\epsilon + \epsilon^2$. Ignore the lower order terms η^2 and ϵ^2 to conclude that $\eta \approx (r_0\epsilon/y_0)$. The upper region has approximate area $r_0\epsilon\delta/y_0$, which implies

$$\begin{aligned}
&\mathbb{P}\{x_0 \le X \le x_0+\delta,\ r_0 \le R \le r_0+\epsilon\} \\
&\quad \approx 2\frac{r_0\epsilon\delta}{y_0} f(x_0,y_0) \\
&\quad \approx \frac{2r_0}{\sqrt{r_0^2-x_0^2}} \frac{\exp(-r_0^2/2)}{2\pi}\epsilon\delta.
\end{aligned}$$

Thus the random variables X and R have joint density

$$\psi(x,r) = \frac{r\exp(-r^2/2)}{\pi\sqrt{r^2-x^2}}\mathbf{1}\{|x|<r,\ 0<r\}.$$

Once again I have omitted the subscript on the dummy variables, to indicate that the argument works for every x, r in the specified range.

For $r > 0$, the random variable R has marginal density

$$\begin{aligned}
g(r) &= \int_{-r}^{r} \psi(x,r)\,dx \\
&= \frac{r\exp(-r^2/2)}{\pi}\int_{-r}^{r}\frac{dx}{\sqrt{r^2-x^2}} \qquad \text{put } x = r\cos\theta \\
&= \frac{r\exp(-r^2/2)}{\pi}\int_{\pi}^{0}\frac{-r\sin\theta}{r\sin\theta}\,d\theta = r\exp(-r^2/2).
\end{aligned}$$

The conditional density for X given $R = r$ equals

$$h(x \mid R=r) = \frac{\psi(x,r)}{g(r)} = \frac{1}{\pi\sqrt{r^2-x^2}} \qquad \text{for } |x|<r \text{ and } r>0.$$

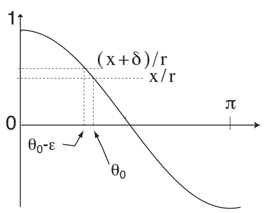
A goodly amount of work.

The calculation is easier when expressed in polar coordinates. From example <11.7> you know how to construct independent $N(0,1)$ distributed random variables by starting with independent random variables $\widetilde{R}$ with density

$$g(r) = r\exp(-r^2/2)\mathbf{1}\{r > 0\},$$

and $U \sim \text{Uniform}(0, 2\pi)$: define $X = \widetilde{R}\cos(U)$ and $Y = \widetilde{R}\sin(U)$.

If we start with X and Y constructed in this way then $R = \sqrt{X^2 + Y^2} = \widetilde{R}$ and the conditional density $h(x \mid R = r)$ is given, for $|x| < r$ by



$$\begin{aligned}
\delta h(x \mid R = r) & \\
&\approx \mathbb{P}\{x \le R\cos(U) \le x + \delta \mid R = r\} \\
&= \mathbb{P}\{x \le r\cos(U) \le x + \delta\} \qquad \text{by independence of } R \text{ and } U \\
&= \mathbb{P}\{\theta_0 - \epsilon \le U \le \theta_0\} + \mathbb{P}\{\theta_0 - \epsilon + \pi \le U \le \theta_0 + \pi\}
\end{aligned}$$

where θ_0 is the unique value in $[0, \pi]$ for which

$$x/r = \cos(\theta_0) \qquad \text{AND} \qquad (x + \delta)/r = \cos(\theta_0 - \epsilon) \approx \cos(\theta_0) + \epsilon\sin(\theta_0).$$

Solve (approximately) for ϵ then substitute into the expression for the conditional density:

$$\delta h(x \mid R = r) \approx \frac{2\epsilon}{2\pi} \approx \frac{\delta}{\pi r\sin(\theta_0)} = \frac{\delta}{\pi r\sqrt{1 - (x/r)^2}}, \qquad \text{for } |x| < r,$$

the same as before. □

<12.2> **Example.** *Let S denote the height (in inches) of a randomly chosen father, and T denote the height (in inches) of his son at maturity. Suppose each of S and T has a $N(\mu, \sigma^2)$ distribution with $\mu = 69$ and $\sigma = 2$. Suppose also that the standardized variables $(S - \mu)/\sigma$ and $(T - \mu)/\sigma$ have a standard bivariate normal distribution with correlation $\rho = .3$.*

If Sam has a height of $S = 74$ inches, what would one predict about the ultimate height T of his young son Tom?

In standardized units, Sam has height $X = (S - \mu)/\sigma$, which we are given to equal 2.5. Tom's ultimate standardized height is $Y = (T - \mu)/\sigma$. By assumption, before the value of X was known, the pair (X, Y) has a standard bivariate normal distribution with correlation ρ. The conditional distribution of Y given that $X = 2.5$ is

$$Y \mid X = 2.5 \sim N(2.5\rho, 1 - \rho^2)$$

In the original units, the conditional distribution of T given $S = 74$ is normal with mean $\mu + 2.5\rho\sigma$ and variance $(1-\rho^2)\sigma^2$, that is,

$$\text{Tom's ultimate height} \mid \text{Sam's height} = 74 \text{ inches} \sim N(70.5, 3.64)$$

If I had to make a guess, I would predict that Tom would ultimately reach a height of 70.5 inches. □

> **Remark.** Notice that Tom expected height (given that Sam is 74 inches) is less than his father's height. This fact is an example of a general phenomenon called "regression towards the mean". The term ***regression***, as a synonym for conditional expectation, has become commonplace in Statistics.

<12.3> **Example.** *Let T_i denote the time to the ith point in a Poisson process with rate λ on $[0,\infty)$. Find the joint distribution of (T_1, T_2) conditional on T_3.*

For fixed $0 < t_1 < t_2 < t_3 < \infty$ and suitably small positive $\delta_1, \delta_2, \delta_3$ define disjoint intervals

$$\begin{aligned} &I_1 = [0, t_1) \quad I_2 = [t_1, t_1 + \delta_1] \quad I_3 = (t_1 + \delta_1, t_2), \\ &I_4 = [t_2, t_2 + \delta_2], \quad I_5 = (t_2 + \delta_2, t_3), \quad I_6 = [t_3, t_3 + \delta_3]. \end{aligned}$$

Write N_j for the number of points landing in I_j, for $j = 1, \dots, 6$. The random variables $N_1, \dots, N_6$ are independent Poissons, with expected values

$$\lambda t_1, \quad \lambda\delta_1, \quad \lambda(t_2 - t_1 - \delta_1), \quad \lambda\delta_2, \quad \lambda(t_3 - t_2 - \delta_2), \quad \lambda\delta_3.$$

To calculate the joint density for (T_1, T_2, T_3) start from

$$\begin{aligned} &\mathbb{P}\{t_1 \le T_1 \le t_1 + \delta_1,\ t_2 \le T_2 \le t_2 + \delta_2, t_3 \le T_3 \le t_3 + \delta_3\} \\ &\quad = \mathbb{P}\{N_1 = 0,\ N_2 = 1,\ N_3 = 0,\ N_4 = 1,\ N_5 = 0,\ N_6 = 1\} \\ &\quad + \text{smaller order terms}. \end{aligned}$$

Here the "smaller order terms" involve probabilities of subsets of events such as $\{N_2 \ge 2,\ N_4 \ge 1,\ N_6 \ge 1\}$, which has very small probability:

$$\mathbb{P}\{N_2 \ge 2\}\mathbb{P}\{N_4 \ge 1\}\mathbb{P}\{N_6 \ge 1\} = o(\delta_1\delta_2\delta_3).$$

Independence also gives a factorization of the main contribution:

$$\begin{aligned} &\mathbb{P}\{N_1 = 0,\ N_2 = 1,\ N_3 = 0,\ N_4 = 1,\ N_5 = 0,\ N_6 = 1\} \\ &\quad = \mathbb{P}\{N_1 = 0\}\mathbb{P}\{N_2 = 1\}\mathbb{P}\{N_3 = 0\}\mathbb{P}\{N_4 = 1\}\mathbb{P}\{N_5 = 0\}\mathbb{P}\{N_6 = 1\} \\ &\quad = e^{-\lambda t_1}[\lambda\delta_1 + o(\delta_1)]e^{-\lambda(t_2 - t_1 - \delta_1)} \times \\ &\qquad\quad \times [\lambda\delta_2 + o(\delta_2)]e^{-\lambda(t_3 - t_2 - \delta_2)}[\lambda\delta_3 + o(\delta_3)] \\ &\quad = \lambda^3\delta_1\delta_2\delta_3\, e^{-\lambda t_3} + o(\delta_1\delta_2\delta_3) \end{aligned}$$

If you think of Δ as a small shoebox (hyperrectangle) with sides δ_1, δ_2, and δ_3, with all three δ_j's of comparable magnitude (you could even take $\delta_1 = \delta_2 = \delta_3$), the preceding calculations reduce to

$$\mathbb{P}\{(T_1, T_2, T_3) \in \Delta\} = (\text{volume of } \Delta)\lambda^3 e^{-\lambda t_3} + \text{smaller order terms}$$

where the "smaller order terms" are small relative to the volume of Δ. Thus the joint density for (T_1, T_2, T_3) is

$$f(t_1, t_2, t_3) = \lambda^3 e^{-\lambda t_3} \mathbb{I}\{0 < t_1 < t_2 < t_3\}.$$

Remark. The indicator function is very important. Without it you would be unpleasantly surprised to find $\iiint_{\mathbb{R}^3} f = \infty$.

Just as a check, calculate the marginal density for T_3 as

$$\begin{aligned} g(t_3) &= \iint_{\mathbb{R}^2} f(t_1, t_2, t_3)\, dt_1\, dt_2 \\ &= \lambda^3 e^{-\lambda t_3} \iint \mathbb{I}\{0 < t_1 < t_2 < t_3\}\, dt_1\, dt_2. \end{aligned}$$

The double integral equals

$$\int \mathbb{I}\{0 < t_2 < t_3\} \left(\int_0^{t_2} 1\, dt_1 \right) = \int_0^{t_3} t_2\, dt_2 = \tfrac{1}{2} t_3^2.$$

That is, T_3 has marginal density

$$g(t_3) = \tfrac{1}{2} \lambda^3 t_3^2 e^{-\lambda t_3} \mathbb{I}\{t_3 > 0\},$$

which agrees with the result calculated in Example <10.1>.

Calculate the conditional density for a given $t_3 > 0$ as

$$\begin{aligned} h(t_1, t_2 \mid T_3 = t_3) &= \frac{f(t_1, t_2, t_3)}{g(t_3)} \\ &= \frac{\lambda^3 e^{-\lambda t_3} \mathbb{I}\{0 < t_1 < t_2 < t_3\}}{\frac{1}{2} \lambda^3 t_3^2 e^{-\lambda t_3}} \\ &= \frac{2}{t_3^2} \mathbb{I}\{0 < t_1 < t_2 < t_3\}. \end{aligned}$$

That is, conditional on $T_3 = t_3$, the pair (T_1, T_2) is uniformly distributed in a triangular region of area $t_3^2/2$. □