Chapter 13

Moment generating functions

13.1 Basic facts

Formally the moment generating function is obtained by substituting $s = e^t$ in the probability generating function.

**Definition.** The moment generating function (m.g.f.) of a random variable $X$ is the function $M_X$ defined by $M_X(t) = \mathbb{E}(e^{Xt})$ for those real $t$ at which the expectation is well defined.

Unfortunately, for some distributions the moment generating function is finite only at $t = 0$. The Cauchy distribution, with density $f(x) = \frac{1}{\pi(1 + x^2)}$ for all $x \in \mathbb{R}$, is an example.

**Remark.** The problem with existence and finiteness is avoided if $t$ is replaced by $it$, where $t$ is real and $i = \sqrt{-1}$. In probability theory the function $\mathbb{E}e^{Xt}$ is usually called the characteristic function, even though the more standard term Fourier transform would cause less confusion.

When the m.g.f. is finite in a neighborhood of the origin it can be expanded in a power series, which gives us some information about the moments (the values of $\mathbb{E}X^k$ for $k = 1, 2, \ldots$) of the distribution:

$$
\mathbb{E}(e^{Xt}) = \sum_{k=0}^{\infty} \frac{\mathbb{E}(Xt)^k}{k!}
$$
The coefficient of \( t^k/k! \) in the series expansion of \( M(t) \) equals the \( k \)th moment, \( \mathbb{E}X^k \).

**Example.** Suppose \( X \) has a standard normal distribution. Its moment generating function equals \( \exp(t^2/2) \), for all real \( t \), because
\[
\int_{-\infty}^{\infty} e^{xt} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left( -\frac{(x-t)^2}{2} + \frac{t^2}{2} \right) dx
\]
\[
= \exp \left( \frac{t^2}{2} \right).
\]

For the last equality, compare with the fact that the \( N(t,1) \) density integrates to 1.

The exponential in \( M_X(t) \) expands to
\[
\sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{t^2}{2} \right)^m = \sum_{m=0}^{\infty} \left( \frac{(2m)!}{m!2^m} \right) \frac{t^{2m}}{(2m)!}
\]
Pick off coefficients.

\[
\mathbb{E}X^2 = \frac{2!}{1!2^1} = 1 \quad \text{(you knew that)}
\]

\[
\mathbb{E}X^4 = \frac{4!}{2!2^2} = 3
\]

\[
\ldots
\]

\[
\mathbb{E}(X^{2m}) = \frac{(2m)!}{m!2^m} \quad \text{for } m \text{ a positive integer.}
\]

The coefficient for each odd power of \( t \) equals zero, which reflects the fact that \( \mathbb{E}X^k = 0 \), by anti-symmetry, if \( k \) is odd.

**Example.** If \( X \sim \text{gamma}(\alpha) \), with \( \alpha > 0 \), then for \( t < 1 \)
\[
M_X(t) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{xt} x^{\alpha-1} e^{-x} dx
\]
\[
= \frac{1}{(1-t)^\alpha \Gamma(\alpha)} \int_0^{\infty} y^{\alpha-1} e^{-y} dy \quad \text{putting } y = (1-t)x
\]
\[
= (1-t)^{-\alpha}.
\]

For \( t \geq 1 \) the integral diverges and \( M_X(t) = \infty \). For \( |t| < 1 \),
\[
M_X(t) = \sum_{k=0}^{\infty} \frac{(-\alpha)}{k} (-t)^k
\]
\[
= \sum_{k=0}^{\infty} (-1)^k (-\alpha)(-\alpha-1)\ldots(-\alpha-k+1) \frac{t^k}{k!}.
\]
The $k$th moment, $E(X^k)$, equals $(\alpha + k - 1)(\alpha + k - 2) \ldots (\alpha)$, the coefficient of $t^k/k!$. Compare with the direct calculation in Example <10.3>.

\[ \square \]

### 13.2 MGF’s determine distributions

If two random variables $X$ and $Y$ have moment generating functions that are finite and equal in some neighborhood of 0 then they have the same distributions. This result is much harder to prove than its analog for probability generating functions.

For example, if $M_X(t) = e^{t^2/2}$, even just for $t$ near 0, then $X$ must have a $N(0, 1)$ distribution.

### 13.3 Approximations via moment generating functions

If $X_n = \xi_1 + \cdots + \xi_n$ with the $\xi_i$’s independently $\text{Ber}(p)$ distributed then

\[
M_{X_n}(t) = E(e^{t\xi_1}e^{t\xi_2} \ldots e^{t\xi_n})
= \left(Ee^{t\xi_1}\right)\left(Ee^{t\xi_2}\right) \ldots \left(Ee^{t\xi_n}\right) \quad \text{by independence}
= (q + pe^t)^n.
\]

That is, the $\text{Bin}(n, p)$ has m.g.f. $(q + pe^t)^n$.

Write $q$ for $1 - p$ and $\sigma_n^2$ for $npq$. You know that the standardized random variable $Z_n := (X_n - np)/\sigma_n$ is approximately $N(0, 1)$ distributed. The moment generating function $M_{Z_n}(t)$ also suggests such an approximation. Then

\[
M_{Z_n}(t) = Ee^{t(X_n - np)/\sigma_n}
= e^{-npt/\sigma}Ee^{X(t/\sigma_n)} = e^{-npt/\sigma}M_{X_n}(t/\sigma_n)
= e^{-npt/\sigma_n}\left(q + pe^t/\sigma_n\right)^n
= \left(qe^{-pt/\sigma_n} + pe^t/\sigma_n\right)^n.
\]

The power series expansion for $q e^{-pt/\sigma} + pe^t/\sigma$ simplifies:

\[
q \left(1 - \frac{pt}{\sigma} + \frac{p^2t^2}{2\sigma^2} - \frac{p^3t^3}{3!\sigma^3} + \ldots\right) + p \left(1 + \frac{qt}{\sigma} + \frac{qt^2}{2!\sigma^2} - \frac{q^3t^3}{3!\sigma^3} + \ldots\right)
= 1 + \frac{pqt}{2\sigma^2} + \frac{pq(p - q)t^3}{6\sigma^3} + \ldots
\]
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For large $n$ use the series expansion $\log(1+z)^n = n(z - z^2/2 + \ldots)$ to deduce that

$$\log M_{Z_n}(t) = \frac{t^2}{2} + \frac{(q-p)t^3}{6\sqrt{npq}} + \text{terms of order } \frac{1}{n} \text{ or smaller}$$

The $t^2/2$ term agree with the logarithm of the moment generating function for the standard normal. As $n$ tends to infinity, the remainder terms tend to zero.

The convergence of $M_{Z_n}(t)$ to $e^{t^2/2}$ can be used to prove rigorously that the distribution of the standardized Binomial “converges to the standard normal” as $n$ tends to infinity. In fact the series expansion for $\log M_n(t)$ is the starting point for a more precise approximation result—but for that story you will have to take the more advanced probability course Statistics 330.