Chapter 13

Moment generating functions

13.1 Basic facts

MGF::overview

Formally the moment generating function is obtained by substituting $s = e^t$ in the probability generating function.

Definition. The moment generating function (m.g.f.) of a random variable X is the function M_X defined by $M_X(t) = \mathbb{E}(e^{Xt})$ for those real t at which the expectation is well defined.

Unfortunately, for some distributions the moment generating function is finite only at $t = 0$. The Cauchy distribution, with density

$$f(x) = \frac{1}{\pi(1+x^2)} \qquad \text{for all } x \in \mathbb{R},$$

is an example.

Remark. The problem with existence and finiteness is avoided if t is replaced by it, where t is real and $i = \sqrt{-1}$. In probability theory the function $\mathbb{E}e^{iXt}$ is usually called the *characteristic function*, even though the more standard term *Fourier transform* would cause less confusion.

When the m.g.f. is finite in a neighborhood of the origin it can be expanded in a power series, which gives us some information about the ***moments*** (the values of $\mathbb{E}X^k$ for $k = 1, 2, \dots$) of the distribution:

$$\mathbb{E}(e^{Xt}) = \sum_{k=0}^{\infty} \frac{\mathbb{E}(Xt)^k}{k!}$$

The coefficient of $t^k/k!$ in the series expansion of $M(t)$ equals the kth moment, $\mathbb{E}X^k$.

normal.mgf <13.1> **Example.** Suppose X has a standard normal distribution. Its moment generating function equals $\exp(t^2/2)$, for all real t, because

$$\int_{-\infty}^{\infty} e^{xt}\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-t)^2}{2}+\frac{t^2}{2}\right)dx$$
$$= \exp\left(\frac{t^2}{2}\right).$$

For the last equality, compare with the fact that the $N(t,1)$ density integrates to 1.

The exponential in $M_X(t)$ expands to

$$\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{t^2}{2}\right)^m = \sum_{m=0}^{\infty}\left(\frac{(2m)!}{m!2^m}\right)\frac{t^{2m}}{(2m)!}$$

Pick off coefficients.

$$\mathbb{E}X^2 = \frac{2!}{1!2^1} = 1 \qquad \text{(you knew that)}$$
$$\mathbb{E}X^4 = \frac{4!}{2!2^2} = 3$$
$$\dots$$
$$\mathbb{E}(X^{2m}) = \frac{(2m)!}{m!2^m} \qquad \text{for } m \text{ a positive integer.}$$

The coefficient for each odd power of t equals zero, which reflects the fact that $\mathbb{E}X^k = 0$, by anti-symmetry, if k is odd.

□

gamma <13.2> **Example.** If $X \sim \text{gamma}(\alpha)$, with $\alpha > 0$, then for $t < 1$

$$M_X(t) = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} e^{xt}x^{\alpha-1}e^{-x}dx$$
$$= \frac{1}{(1-t)^{\alpha}\Gamma(\alpha)}\int_0^{\infty} y^{\alpha-1}e^{-y}dy \qquad \text{putting } y=(1-t)x$$
$$= (1-t)^{-\alpha}.$$

For $t \geq 1$ the integral diverges and $M_X(t) = \infty$. For $|t| < 1$,

$$M_X(t) = \sum\nolimits_{k=0}^{\infty}\binom{-\alpha}{k}(-t)^k$$
$$= \sum\nolimits_{k=0}^{\infty}(-1)^k\frac{(-\alpha)(-\alpha-1)\dots(-\alpha-k+1)}{k!}t^k.$$

The kth moment, $\mathbb{E}(X^k)$, equals $(\alpha+k-1))(\alpha+k-2)\dots(\alpha)$, the coeeficient of $t^k/k!$. Compare with the direct calculation in Example <10.3>.

$\square$

13.2 MGF's determine distributions

MGF::uniqueness

If two random variables X and Y have moment generating functions that are finite and equal in some neighborhood of 0 then they have the same distributions. This result is much harder to prove than its analog for probability generating functions.

For example, if $M_X(t) = e^{t^2/2}$, even just for t near 0, then X must have a $N(0,1)$ distribution.

13.3 Approximations via moment generating functions

MGF::cty

If $X_n = \xi_1 + \dots + \xi_n$ with the ξ_i's independently Ber(p) distributed then

$$
\begin{aligned}
M_{X_n}(t) &= \mathbb{E}\left(e^{t\xi_1}e^{t\xi_2}\dots e^{t\xi_n}\right)\\
&= \left(\mathbb{E}e^{t\xi_1}\right)\left(\mathbb{E}e^{t\xi_2}\right)\dots\left(\mathbb{E}e^{t\xi_n}\right) \qquad \text{by independence}\\
&= \left(q+pe^t\right)^n.
\end{aligned}
$$

That is, the Bin(n,p) has m.g.f. $\left(q+pe^t\right)^n$.

Write q for $1-p$ and σ_n^2 for npq. You know that the standardized random variable $Z_n := (X_n - np)/\sigma_n$ is approximately $N(0,1)$ distributed. The moment generating function $M_{Z_n}(t)$ also suggests such an approximation. Then

$$
\begin{aligned}
M_{Z_n}(t) &= \mathbb{E}e^{t(X-np)/\sigma_n}\\
&= e^{-npt/\sigma}\mathbb{E}e^{X(t/\sigma_n)} = e^{-npt/\sigma}M_{X_n}(t/\sigma_n)\\
&= e^{-npt/\sigma_n}\left(q+pe^{t/\sigma_n}\right)^n\\
&= \left(qe^{-pt/\sigma_n}+pe^{qt/\sigma_n}\right)^n.
\end{aligned}
$$

The power series expansion for $qe^{-pt/\sigma}+pe^{qt/\sigma}$ simplifies:

$$
\begin{aligned}
&q\left(1-\frac{pt}{\sigma}+\frac{p^2t^2}{2!\sigma^2}-\frac{p^3t^3}{3!\sigma^3}+\dots\right)+p\left(1+\frac{qt}{\sigma}+\frac{q^2t^2}{2!\sigma^2}-\frac{q^3t^3}{3!\sigma^3}+\dots\right)\\
&= 1+\frac{pqt}{2\sigma^2}+\frac{pq(p-q)t^3}{6\sigma^3}+\dots
\end{aligned}
$$

For large n use the series expansion $\log(1+z)^n = n(z-z^2/2+\dots)$ to deduce that

$$\log M_{Z_n}(t) = \frac{t^2}{2} + \frac{(q-p)t^3}{6\sqrt{npq}} + \text{ terms of order } \frac{1}{n} \text{ or smaller}$$

The $t^2/2$ term agree with the logarithm of the moment generating function for the standard normal. As n tends to infinity, the remainder terms tend to zero.

The convergence of $M_{Z_n}(t)$ to $e^{t^2/2}$ can be used to prove rigorously that the distribution of the standardized Binomial "converges to the standard normal" as n tends to infinity. In fact the series expansion for $\log M_n(t)$ is the starting point for a more precise approximation result—but for that story you will have to take the more advanced probability course Statistics 330.