Chapter 11

Conditional densities

Density functions determine continuous distributions. If the distribution is conditional on some information, then the density is called a conditional density. When the conditioning information involves a random variable with a continuous distribution, the calculation of the conditional density involves arguments like those of Chapter 10.

To illustrate a few of the possible methods for dealing with conditional densities, this Chapter will provide three solutions to the following problem:

Let X and Y be independent random variables, each distributed $N(0,1)$. For each $r > 0$, find the conditional distribution of X given $\sqrt{X^2+Y^2} = r$.

Ratios of small probabilities

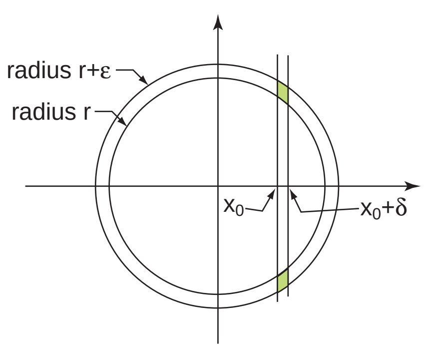
The joint density for (X, Y) equals

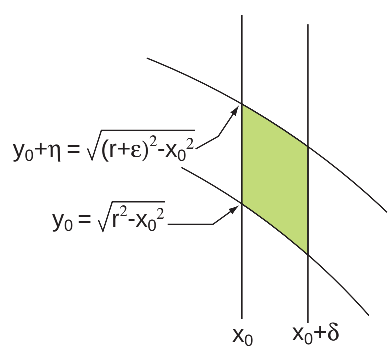
$$f(x,y) = \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right)$$

Write R for $\sqrt{X^2+Y^2}$. For a fixed x_0 and small positive δ,

<11.1>
$$\begin{aligned}\mathbb{P}\{x_0 \le X \le x_0+\delta \mid R = r\} &\approx \mathbb{P}\{x_0 \le X \le x_0+\delta \mid r \le R \le r+\epsilon\} \\ &= \frac{\mathbb{P}\{x_0 \le X \le x_0+\delta, r \le R \le r+\epsilon\}}{\mathbb{P}\{r \le R \le r+\epsilon\}}\end{aligned}$$

Consider the probability in the numerator. For $|x_0| < r$, the event corresponds to the two small regions in the (X,Y)-plane lying between the lines $x = x_0$ and $x = x_0 + \delta$, and between the circles centered at the origin with radii r and $r + \epsilon$.





By symmetry, both regions contribute the same probability. Consider the upper region. For small δ and ϵ, the region is approximately a parallelogram, with base

$$\eta = \sqrt{(r+\epsilon)^2 - x_0^2} - \sqrt{r^2 - x_0^2}$$

and width δ. We could expand the expression for η as a power series in ϵ by multiple applications of Taylor's theorem. Equivalently, use the fact that

$$(y_0+\eta)^2 = (r+\epsilon)^2 - x_0^2 \qquad \text{where } y_0^2 = r^2 - x_0^2$$

Expand each square, discarding terms (η^2 and ϵ^2) of smaller order, leaving

$$y_0^2 + 2\eta y_0 \approx r^2 + 2r\epsilon - x_0^2$$

or $\eta \approx (r\epsilon/y_0)$. The upper region has approximate area $r\epsilon\delta/y_0$. The numerator in <11.1> equals

$$2\frac{r\epsilon\delta}{y_0} f(x_0, y_0) + \text{ smaller order terms} \approx \epsilon\delta \frac{2r}{\sqrt{r^2-x_0^2}} \frac{\exp(-r^2/2)}{2\pi}$$

The denominator in <11.1> could be calculated by a similar argument, but it is not really necessary. If R has density $g(\cdot)$ the probability in the denominator equals

$$\epsilon g(r) + \text{ smaller order terms}$$

which gives

$$\epsilon\delta \frac{2r}{\sqrt{r^2-x_0^2}} \frac{\exp(-r^2/2)}{2\pi} \Big/ (\epsilon g(r))$$

as the approximation for the ratio in <11.1>. The ϵ cancels, leaving

$$\frac{r\exp(-r^2/2)}{\pi g(r)} \frac{1}{\sqrt{r^2-x^2}} \qquad \text{for } |x| < r$$

as the conditional density, which I denote by $f(x \mid R = r)$. (Once again I have omitted the subscript on the x_0, to indicate that the argument works for every x in the range.) The function of r out front plays the role of the constant (for fixed r) to ensure that

$$\int_{-r}^{r} f(x \mid R = r)\,dx = 1 \qquad \text{for each } r > 0.$$

We can calculate the necessary scaling constant directly, using the fact

$$\int_{-r}^{r} \frac{dx}{\sqrt{r^2-x^2}} = 2\int_0^{\pi/2} \frac{r\cos\theta\,d\theta}{\sqrt{r^2 - r^2\sin^2\theta}} = \pi$$

Thus $r\exp(-r^2/2)/g(r) = 1$, and

$$f(x \mid R = r) = \frac{1}{\pi\sqrt{r^2-x^2}} \qquad \text{for } |x| < r$$

as the density for the conditional distribution of X given $R = r$.

The conditional distribution for the random variable $T = X/R$ given $R = r$ is even simpler. It has density

<11.2> $$r f(rt \mid R = r) = \frac{1}{\pi\sqrt{1-t^2}} \qquad \text{for } |t| < 1$$

As a bonus we get $g(r) = r\exp(-r^2/2)$ for $r > 0$. A change-of-variable argument (compare with Problem 10.1 or the discussion in Chapter 9 regarding transformations for univariate densities) that $R^2/2$ has a standard exponential distribution.

Ratios of densities

The calculations of probabilities for (X, Y) lying in small regions shaped almost like parallelograms should have reminded you of the calculations in Chapter 10, for calculating joint densities for functions of random variables with jointly continuous distributions. Indeed, the approximation

$$\mathbb{P}\{x_0 \le X \le x_0 + \delta, r \le R \le r + \epsilon\} \approx \epsilon\delta \frac{2r}{\sqrt{r^2 - x_0^2}} \frac{\exp(-r^2/2)}{2\pi}$$

is exactly the result needed to show that the random variables X and R have a jointly continuous distribution with joint density

$$\psi(x,r) = \frac{r}{\sqrt{r^2 - x_0^2}} \frac{\exp(-r^2/2)}{\pi} \qquad \text{for } 0 < |x| < r$$

The marginal density g for R is given by the integral

$$\int_{-\infty}^{\infty} \psi(x,r)\,dx = \int_{-r}^{r} \frac{r}{\sqrt{r^2 - x_0^2}} \frac{\exp(-r^2/2)}{\pi}\,dx = r\exp(-r^2/2)$$

The conditional density $f(\cdot \mid R = r)$ was just the ratio ψ/g:

$$\text{conditional density } f(x \mid R = r) = \frac{\text{joint density } \psi(x,r)}{\text{marginal density } g(r)}$$

An analogous formula (with appropriate joint and marginal densities) works for the calculation of conditional densities in general.

Transformation to new random variables

The original problem can be recast in a much simpler form when X and Y are written in polar coordinates: $X = R\cos(\Theta)$ and $Y = R\sin(\Theta)$. The distribution of X/R given $R = r$ is just the conditional distribution of $\cos(\Theta)$. Formula <11.2> shows that this conditonal distribution does not depend on r, that is, $\cos(\Theta)$ is independent of R. The formula suggests that Θ and R are independent random variables.

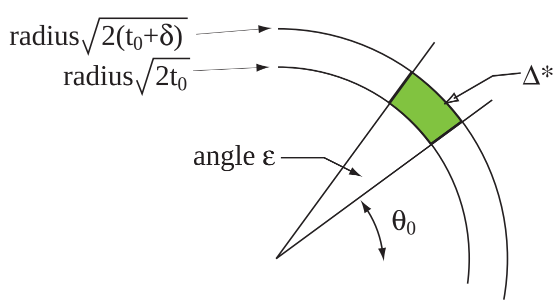
To verify the suggested independence we can work backwards.

<11.3> **Exercise.** Suppose W has a standard exponential distribution independent of Θ, which is uniformly distributed on $[0, 2\pi]$. Put $R = \sqrt{2W}$. Show that the random variables

$$X = R\cos\Theta$$
$$Y = R\sin\Theta$$

are independent and each is $N(0,1)$ distributed.

SOLUTION: The rectangle Δ with corners (t_0, θ_0), and $(t_0 + \delta, \theta_0 + \epsilon)$ in the (W, Θ) strip corresponds to a region Δ^* in the (X, Y)-plane bounded by radial lines at angles θ_0 and $\theta_0 + \epsilon$ from the X-axis and two circles, of radii $\sqrt{2t_0}$ and $\sqrt{2(t_0 + \delta)}$, centered at the origin.



Simple geometry will give the area of Δ^*. (You might calculate the Jacobian as a cross-check.) The annular region between the two circles has area $\pi 2(t_0 + \delta) - \pi(2t_0)$. The two radial lines carve out a proportion $\epsilon/(2\pi)$ of that area:

$$\text{area of } \Delta^* = \frac{\epsilon}{2\pi} 2\pi\delta = \epsilon\delta$$

The joint density $f(x, y)$ for (X, Y) at the point $(x_0, y_0) = (\sqrt{2t_0}\cos\theta_0, \sqrt{2t_0}\sin\theta_0)$ is given by

$$\begin{aligned}\epsilon\delta f(x_0, y_0) &\approx \mathbb{P}\{(X, Y) \in \Delta^*\}\\ &= \mathbb{P}\{\theta_0 \le \Theta \le \theta_0 + \epsilon, t_0 \le W \le t_0 + \delta\}\\ &= \mathbb{P}\{\theta_0 \le \Theta \le \theta_0 + \epsilon\}\mathbb{P}\{t_0 \le W \le t_0 + \delta\} \qquad \text{by independence}\\ &\approx \frac{\epsilon}{2\pi}\delta\exp(-t_0) \qquad \text{where } t_0 = \frac{x_0^2 + y_0^2}{2}\end{aligned}$$

That is

$$f(x, y) = \frac{1}{2\pi}\exp\left(-\frac{x^2 + y^2}{2}\right).$$

The random variables X, and Y have the joint density of a pair of independent $N(0, 1)$ distributed variates. □

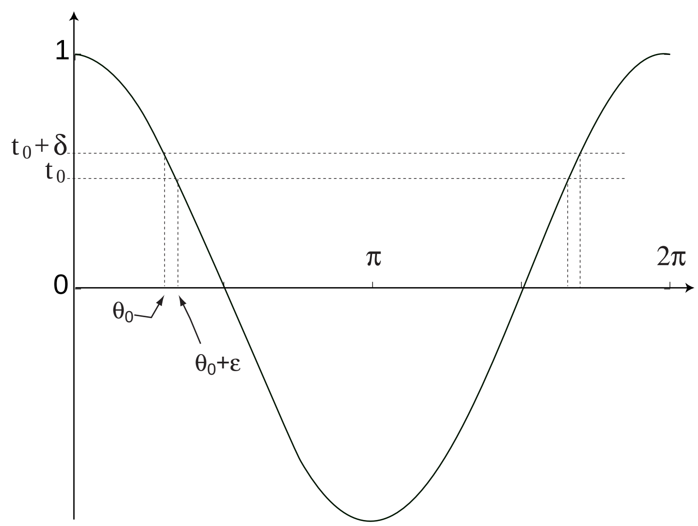
In the motivating problem for this Chapter, we could have taken X and Y as in the previous Exercise. Then $X^2 + Y^2 = 2W$, and the problem asks for the conditional distribution of $\sqrt{2W}\cos(\Theta)$ given that $W = r^2/2$. The conditioning lets us put $\sqrt{2W}$ equal to the constant r. The independence of W and Θ lets us ignore the effects on $\cos(\Theta)$ of the conditioning; the conditional density for $\cos(\Theta)$ is the same as its marginal density. To calculate that marginal density, take $-1 < t_0 < 1$ and small $\delta > 0$. Then

$$\mathbb{P}\{t_0 \le \cos(\Theta) \le t_0 + \delta\} = 2\mathbb{P}\{\theta_0 \le \Theta \le \theta_0 + \epsilon\} = \frac{2\epsilon}{2\pi},$$

where θ_0 and $\theta_0 + \epsilon$ are the values in $(0, \pi)$ for which $\cos(\theta_0) = t_0$ and $\cos(\theta_0 + \epsilon) = t_0 + \delta$. Arguing that $\delta/\epsilon \approx \sin(\theta_0)$, we deduce that the distribution for $\cos(\Theta)$ has density

$$\frac{1}{\pi\sqrt{1 - t^2}} \qquad \text{for } |t| < 1,$$

as in <11.2>.



Remarks: The Box-Muller method generates independent $N(0, 1)$ variates X_1 and X_2, from two independent Uniform(0, 1) variates, U_1 and U_2, by

$$X_1 = \sqrt{-2\log U_1}\cos(2\pi U_2)$$
$$X_2 = \sqrt{-2\log U_1}\sin(2\pi U_2)$$

□ Why does the method work?