Fair prices: appendix to Chapter 2

Probability could be studied purely as a piece of mathematics, divorced from any interpretation, but then one would lose much of the intuition that accompanies the interpretation. The most widely accepted view interprets probabilities and expectations as long run averages, anticipating the formal laws of large numbers that make precise a sense in which averages should settle down to expectations over a long sequence of independent trials. As an aid to intuition I prefer another interpretation, which does not depend on a prelimininary concept of independence, and which concentrates attention on the linearity properties of expectations.

Consider a situation—a bet if you will—where you stand to receive an uncertain return X. You could think of X as a random variable, a real-valued function on a sample space S. For the moment forget about any probability measure on S. Suppose you consider $p(X)$ the fair price to pay in order to receive X. What properties must $p(\cdot)$ have?

•fair bet

Your net return will be the random quantity $X'(\omega) = X(\omega) - p(X)$. Call the random variable X' a FAIR BET. Unless you start worrying about utilities you should find the following properties reasonable.

•fair + fair = fair

(i) FAIR + FAIR = FAIR. That is, if you consider $p(X)$ fair for X and $p(Y)$ fair for Y then you should be prepared to make both bets, paying $p(X) + p(Y)$ to receive $X + Y$.

•constant × fair = fair

(ii) CONSTANT × FAIR = FAIR. That is, you shouldn't object if I suggest you pay $2p(X)$ to receive $2X$ (actually, that particular example is a special case of (i)) or $3.76p(X)$ to receive $3.76X$, or $-p(X)$ to receive $-X$. The last example corresponds to willingness to take either side of a fair bet. In general, to receive cX you should pay $cp(X)$, for constant c.

Properties (i) and (ii) imply that the collection of all fair bets is a vector space.

•coherency

There is a third reasonable property that goes by several names: COHERENCY or NONEXISTENCE OF A DUTCH BOOK, the NO-ARBITRAGE REQUIREMENT, or the NO-FREE-LUNCH PRINCIPLE:

(iii) There is no fair return X' for which $X'(\omega) \geq 0$ for all ω with strict inequality for at least one ω.

If you were to declare such an X' to be fair I would be delighted to offer you the opportunity to receive a net return of $-10^{100}X'$. I couldn't lose.

<1> **Lemma.** *Properties (i), (ii), and (iii) imply that $p(\cdot)$ is an increasing linear functional on random variables.*

Proof. For constants α and β, and random variables X and Y with fair prices $p(X)$ and $p(Y)$, consider the combined effect of the following fair bets:

you pay me $\alpha p(X)$ to receive αX

you pay me $\beta p(Y)$ to receive βY

I pay you $p(\alpha X + \beta Y)$ to receive $(\alpha X + \beta Y)$.

Your net return is a constant,

$$c = p(\alpha X + \beta Y) - \alpha p(X) - \beta p(Y).$$

If $c > 0$ you violate (iii); if $c < 0$ take the other side of the bet to violate (iii). That proves linearity.

To prove that $p(\cdot)$ is increasing, suppose $X(\omega) \geq Y(\omega)$ for all ω. If you claim that $p(X) < p(Y)$ then I would be happy for you to accept the bet that delivers

$$(Y - p(Y)) - (X - p(X)) = -(X - Y) - (p(Y) - p(X)),$$

□ which is always < 0.

<2> **Corollary.** *A random return X is fair if and only if it has a zero fair price: $p(X) = 0$.*

As a special case, consider the bet that returns 1 if an event F occurs, and 0 otherwise. If you identify the event F with the random variable taking the value 1 on F and 0 on F^c (that is, the indicator of the event F), then it follows directly from Lemma <.1> that $p(\cdot)$ is additive: $p(F_1 \cup F_2) = p(F_1) + p(F_2)$ for disjoint events F_1 and F_2. That is, p defines a finitely additive set-function on events. As an exercise you might show that $p(\emptyset) = 0$ and $p(S) = 1$. The set function $p(\cdot)$ has most of the properties required of a probability measure.

Contingent bets

•contingent

Things become much more interesting if you are prepared to make a bet to receive an amount X, but only when some event F occurs. That is, the bet is made CONTINGENT on the occurrence of F. Typically, knowledge of the occurrence of F should change the fair price, which we could denote by $p(X \mid F)$. Let me write Z for the indicator function of the event F, that is,

$$Z = \begin{cases} 1 & \text{if event } F \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Then the net return from the contingent bet is $(X - p(X \mid F))\, Z$, which is fair. The indicator function Z ensures that money changes hands only when F occurs. By linearity of $p(\cdot)$, it follows that

$$0 = p\left(XZ - p(X \mid F)Z\right) = p(XZ) - p(X \mid F)p(F) \tag{$*$}$$

Multiple appeals to this identity generate rule E4 for expectations: *If S is partitioned into disjoint events $F_1, \ldots, F_k$, then*

$$p(X) = \sum_{i=1}^{k} p(F_i)p(X \mid F_i).$$

To verify the assertion, write Z_i for the indicator function of F_i. Notice that $\sum_i Z_i = 1$. Then

$$\begin{aligned} p(X) = p\left(\sum_i XZ_i\right) &= \sum_i p\left(XZ_i\right) \\ &= \sum_i p(X \mid F_i)p(F_i) \qquad \text{by } (*), \end{aligned}$$

as asserted.

If you rewrite $p(X)$ as the expected value $\mathbb{E}X$, and $p(F)$ as $\mathbb{P}F$, you will have an example of a useful probability calculation based on linearity properties of expectations. Perhaps you wonder why we should use two different symbols for the price attached to the random return, depending on whether it takes only values 0 and 1 or not. If so, you have seen one of the virtues of the linear functional notation that is sometimes adopted in more mathematical works on probability theory

If you already knew about the possibility of infinite expectations, you would have realized that I should have imposed some restrictions on the class of random variables for which fair prices were defined, if I were seriously trying to construct a rigorous system of axioms.

See Bruno de Finetti, *Theory of Probability*, Vol. 1, (Wiley, New York), for a detailed discussion of expectations as fair prices.