Brownian motion

Let $X = \{X_t : t \in \mathbb{R}^+\}$ be a real-valued stochastic process: a familty of real random variables all defined on the same probability space Ω. Define

$$\begin{aligned}\mathcal{F}_t &= \text{"information available by observing the process up to time } t\text{"}\\ &= \text{what we learn by observing } X_s \text{ for } 0 \le s \le t\end{aligned}$$

- Call X a standard Brownian motion if
  - (i) $X(,\omega)$ is a continuous function on $\mathbb{R}^+$, for each fixed ω
  - (ii) $X(0,\omega) = 0$ for all ω
  - (iii) for each $s \ge 0$,
    $$\{X_t - X_s : t \ge s\} \qquad \text{is independent of} \qquad \mathcal{F}_s$$
  - (iv) $X_t - X_s$ is $N(0, t-s)$ distributed for each $0 \le s < t$
- Another way to express (iii):
  - (iii)′ $X_t - X_s \mid \mathcal{F}_s \sim N(0, t-s)$ for $s < t$.
- Equivalent way to express (iii) and (iv): for each $0 \le t_1 \le t_2 \le \ldots \le t_k$, the random vector $(X_{t_1}, \ldots, X_{t_k})$ has a multivariate normal distribution with zero means and covariances given by

$$\operatorname{cov}(X_s, X_t) = \min(s,t)$$

Useful facts.

For a fixed $\tau \ge 0$ define

$$Z_t = X_{\tau+t} - X_\tau \qquad \text{for } t \ge 0$$

Markov property: Z is a Brownian motion independent of $\mathcal{F}_\tau$ = information available up to time τ.

Strong Markov property: Same assertion holds for stopping times τ.

Time reversal: Define $Z_t = tX_{1/t}$ for $t > 0$, with $Z_0 = 0$. Then $\{Z_t : t \in \mathbb{R}^+\}$ is a also a Brownian motion.

Martingale properties:

- Abbreviate $\mathbb{E}(\ldots \mid \mathcal{F}_t)$to $\mathbb{E}_t(\ldots)$
- The Brownian motion process is a martingale: for $s < t$,

$$\mathbb{E}_s(X_t) = \mathbb{E}_s(X_s) + \mathbb{E}_s(X_t - X_s) = X_s \qquad \text{by (iii)'}.$$

- The process $M_t = X_t^2 - t$ is a martingale: for $s < t$,

$$\begin{aligned}\mathbb{E}_s(M_t) &= \mathbb{E}_s\left(X_s + \Delta X\right)^2 - t \qquad \text{where } \Delta X := X_t - X_s\\ &= X_s^2 + 2X_s\mathbb{E}_s(\Delta X) + \mathbb{E}_s(\Delta X)^2 - t\\ &= M_s \qquad \text{because } \mathbb{E}_s(\Delta X) = 0 \text{ and } \mathbb{E}_s(\Delta X)^2 = t - s.\end{aligned}$$

- For each real θ, the process $Y_t = \exp\left(\theta X_t - \frac{1}{2}\theta^2 t\right)$ is a martingale: for $s < t$,

$$\begin{aligned}\mathbb{E}_s(Y_t) &= \mathbb{E}_s\left(Y_s \exp(\theta\Delta X - \tfrac{1}{2}\theta^2(t-s)\right)\\ &= Y_s\mathbb{E}_s e^{\theta\Delta X}\exp\left(-\tfrac{1}{2}\theta^2(t-s)\right)\\ &= Y_s \qquad \text{because } \Delta X \mid \mathcal{F}_s \sim N(0, t-s).\end{aligned}$$

Suppose $\{X_t : 0 \le t \le 1\}$ a martingale with continuous sample paths and $X_0 = 0$. Suppose also that $X_t^2 - t$ is a martingale. Then X is a Brownian motion.

Heuristics. I'll give a rough proof for why X_1 is $N(0,1)$ distributed.

Let $f(x,t)$ be a smooth function of two arguments, $x \in \mathbb{R}$ and $t \in [0,1]$. Define

$$f_x = \frac{\partial f}{\partial x} \qquad \text{and} \qquad f_{xx} = \frac{\partial^2 f}{\partial^2 x} \qquad \text{and} \qquad f_t = \frac{\partial f}{\partial t}.$$

Let $h = 1/n$ for some large positive integer n. Define $t_i = ih$ for $i = 0, 1, \ldots, n$. Write $\Delta_i X$ for $X(t_i + h) - X(t_i)$. Then

$$\begin{aligned}\mathbb{E} f(X_1, 1) - \mathbb{E} f(X_0, 0) &= \sum_{i<n} \big(\mathbb{E} f(X_{t_i+h}, t_i + h) - \mathbb{E} f(X_{t_i}, t_i)\big) \\ &\approx \sum_{i<n} \mathbb{E}\left((\Delta_i X) f_x(X_{t_i}, t_i) + \tfrac{1}{2}(\Delta_i X)^2 f_{xx}(X_{t_i}, t_i) + h f_t(X_{t_i}, t_i)\right)\end{aligned}$$

Independence of $\Delta_i X$ from $\mathcal{F}_{t_i}$ gives a factorization for the ith sumand:

$$\mathbb{E}(\Delta_i X)\mathbb{E} f_x(X_{t_i}, t_i) + \tfrac{1}{2}\mathbb{E}(\Delta_i X)^2 \mathbb{E} f_{xx}(X_{t_i}, t_i) = 0 + \tfrac{1}{2} h \mathbb{E} f_{xx}(X_{t_i}, t_i)$$

The sum then takes the form of an approximating sum for the integral

$$\int_0^1 \left(\tfrac{1}{2}\mathbb{E} f_{xx}(X_s, s) + \mathbb{E} f_t(X_s, s)\right) ds$$

If we paid more attention to the errors of approximation we would see that their contributions go to zero as the $\{t_i\}$ grid gets finer. In the limit we have

$$\mathbb{E} f(X_1, 1) - \mathbb{E} f(X_0, 0) = \mathbb{E} \int_0^t \left(\tfrac{1}{2} f_{xx}(X_s, s) + f_t(X_s, s)\right) ds$$

Now specialize to the case $f(x,s) = \exp\left(\theta x - \tfrac{1}{2}\theta^2 s\right)$, with θ a fixed real constant. By direct calculation, we have

$$f_x = \theta f(x,s) \qquad \text{and} \qquad f_{xx} = \theta^2 f(x,s) \qquad \text{and} \qquad f_t = -\tfrac{1}{2}\theta^2 f(x,s)$$

Thus

$$\mathbb{E} e^{\theta X_1} e^{-\theta^2/2} - 1 = \int_0^1 0 \, ds = 0.$$

That is, X_1 has the moment generating function $\exp(\theta^2/2)$, which identifies it as having a $N(0,1)$ distribution.

How to build a Brownian motion

- In Euclidean space with $e_1, \ldots, e_N$ orthogonal unit vectors, if $z \in \text{span}\{e_1, \ldots, e_N\}$ then

$$z = \sum\nolimits_{i \le N} \alpha_i e_i \qquad \text{with } \alpha_i = \langle z, e_i \rangle \quad \text{(inner product)}$$

- For real-valued functions f and g on $(0,1]$ define

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx \qquad \text{and} \qquad \|f\| = \sqrt{\langle f, f \rangle}.$$

- Let $e_1, \ldots, e_N$ be real-valued functions on $(0,1]$ with

$$\langle e_i, e_j \rangle := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

If $f, g \in \mathcal{E}_N := \text{span}\{e_1, \dots, e_N\}$ then

$$f(x) = \sum\nolimits_{i \le N} \alpha_i e_i(x) \qquad \text{with } \alpha_i = \langle f, e_i \rangle$$
$$g(x) = \sum\nolimits_{i \le N} \beta_i e_i(x) \qquad \text{with } \beta_i = \langle g, e_i \rangle$$

Consequence:

$$\langle f, g \rangle = \sum\nolimits_{i,j} \alpha_i \beta_j \langle e_i, e_j \rangle = \sum\nolimits_{i \le N} \alpha_i \beta_i = \sum\nolimits_{i \le N} \langle f, e_i \rangle \langle g, e_i \rangle$$

RANDOM COEFFICIENTS.

Write $\mathcal{L}^2$ for the set of functions f on $(0, 1]$ with $\int_0^1 f(x)^2 dx < \infty$.

- Let $\eta_1, \eta_2, \dots$ be independent, each $N(0, 1)$ distributed. For each f in $\mathcal{L}^2$ define

$$Z_N(f) := \sum\nolimits_{i \le N} \langle f, e_i \rangle \eta_i$$

- The random variables $\{Z_N(f) : f \in \mathcal{L}^2\}$ have a joint normal distribution with zero means and covariances given by

$$\text{cov}(Z_N(f), Z_N(g)) = \sum\nolimits_{i,j} \langle f, e_i \rangle \text{cov}(\eta_i, \eta_j) \langle g, e_j \rangle = \sum\nolimits_{i \le N} \langle f, e_i \rangle \langle g, e_i \rangle$$

In particular, if $f, g \in \mathcal{E}_N$ then

$$\text{cov}(Z_N(f), Z_N(g)) = \langle f, g \rangle \qquad \text{and} \qquad Z_N(f) \sim N(0, \|f\|^2).$$

- For $0 \le t \le 1$ define $f_t(x) = \begin{cases} 1 & \text{if } x \le t \\ 0 & \text{otherwise} \end{cases}$. Then

$$\langle f_s, f_t \rangle = \int_0^1 1\{x \le s\}\, 1\{x \le t\}\, dx = \min(s, t).$$

- If $f_t \in \mathcal{E}_N$ then $Z_N(f_t) \sim N(0, t)$
- If $f_t, f_s \in \mathcal{E}_N$ then $\text{cov}(Z_N(f_s), Z_N(f_t)) = \min(s, t)$.
- Let N tend to ∞. Convergence? In the limit (if it exists) we have a Gaussian stochastic process $\{Z(f_t) : 0 \le t \le 1\}$ with the means and variances desired for Brownian motion.

CONTINUITY OF SAMPLE PATHS.

- For $k = 0, 1, \dots$ and $0 \le i < 2^k$ define functions on $(0, 1]$ by

$$H_{i,k}(x) = 1\{i2^{-k} < x \le (i + 1/2)2^{-k}\} - 1\{(i + 1/2)2^{-k} < x \le (i + 1)2^{-k}\}$$

Note that $|H_{i,k}|$ is the indicator function of the interval $J_{i,k} = (i/2^k, (i + 1)/2^k]$ and

$$\int_0^1 H_{i,k}(x)^2\, dx = \int \{x \in J_{i,k}\}\, dx = 2^{-k}.$$

- (Haar basis) The functions $e_{i,k}(x) = 2^{k/2} H_{i,k}(x)^2$ satisfy

$$\langle e_{i,k}, e_{i',k'} \rangle = \begin{cases} 1 & \text{if } i = i' \text{ and } k = k' \\ 0 & \text{otherwise} \end{cases}$$

Moreover, each $e_{i,k}$ is orthogonal to the constant function $U(x) \equiv 1$, in the sense that $\langle e_{i,k}, U \rangle = 0$.

- Redefine $\mathcal{E}_N$ to be

$$\mathcal{E}_N = \text{span}\left(\{U\} \cup \{e_{i,k} : 0 \le i < 2^k,\ k = 0, 1, \dots, N\}\right)$$

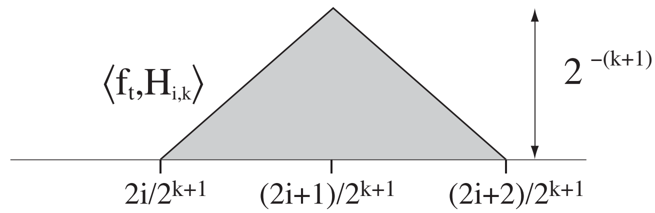
It is not hard to show that $\mathcal{E}_N$ consists of all those real-valued functions on $(0, 1]$ that take a constant value on each $J_{i,k}$ subinterval.

- Let η and $\eta_{i,k}$ for $0 \le i < 2^k$ and $k = 0, 1, \dots$ be independent $N(0, 1)$ random variables. Using the Haar basis functions we have, for $0 \le t \le 1$,

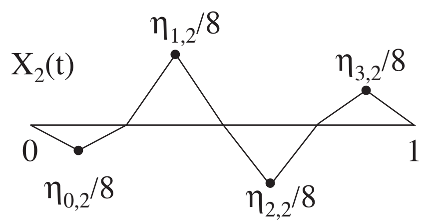
$$Z_N(f_t) = \langle f_t, U \rangle \eta + \sum\nolimits_{k=0}^{N} \sum\nolimits_{0 \le i < 2^k} \langle f_t, e_{i,k} \rangle \eta_{i,k}$$

<1>
$$= t\eta + \sum\nolimits_{k=0}^{N} 2^{k/2} X_k(t) \qquad \text{where } X_k(t) = \sum\nolimits_{0 \le i < 2^k} \langle f_t, H_{i,k} \rangle \eta_{i,k}$$

- As a function of t, each $\langle f_t, H_{i,k}\rangle$ is nonzero only in the interval $J_{i,k}$, within which it is piecewise linear, achieving its maximum value of $2^{-(k+1)}$ at the midpoint, $(2i+1)/2^{k+1}$:



- The process $X_k(t)$ has continuous, piecewise linear sample paths. It takes the value 0 at $t = i/2^k$ for $i = 0, 1, \ldots, 2^k$. It takes the value $\eta_{i,k}/2^{k+1}$ at the point $(2i+1)/2^{k+1}$.



- Thus $\max_t |X_k(t)| = 2^{-(k+1)} \max_i |\eta_{i,k}|$.
- From HW sheet 5,
$$\mathbb{E}\max_t |X_k(t)| = 2^{-(k+1)}\mathbb{E}\max_i |\eta_{i,k}| \le 2^{-(k+1)}\sqrt{2\log(2^k)}$$
and hence
$$\mathbb{E}\sum_{k=0}^{\infty} 2^{k/2}\max_t |X_k(t)| \le \sum_{k=0}^{\infty} 2^{k/2} 2^{-(k+1)}\sqrt{2\log(2^k)} < \infty.$$
The random variable
$$\sum_{k=0}^{\infty} 2^{k/2}\max_t |X_k(t)|$$
has a finite expectation. It must be finite everywhere, except possibly on a set Ω_0^c with zero probability. For all sample points ω in Ω_0, the sum in <1> converges uniformly in t. The sample paths of the limit process, being uniform limits of functions continuous in t, are continuous, at least for ω in Ω_0.
- Fix sample paths for $\omega \in \Omega_0^c$?

Total variation and quadratic variation of a function

- Let f be a real-valued function on $[0, 1]$. Define the total variation of f by
$$\mathcal{V}(f) = \sup_{\mathbb{G}} \sum_i |f(t_{i+1}) - f(t_i)|,$$
where $\mathbb{G}$ ranges over all finite grids $0 = t_0 < t_1, \ldots < t_k = 1$. Say that f is of bounded variation if $\mathcal{V}(f) < \infty$.
- Define quadratic variation of f for a grid $\mathbb{G}$ as
$$\mathcal{Q}(f, \mathbb{G}) = \sum_i |f(t_{i+1}) - f(t_i)|^2 \qquad \text{where } \mathbb{G} \text{ is the grid } 0 = t_0 < t_1, \ldots < t_k = 1.$$
- Fact: If f is continuous and has bounded variation then $\mathcal{Q}(f, \mathbb{G}) \to 0$ as $\text{mesh}(\mathbb{G}) \to 0$, where $\text{mesh}(\mathbb{G}) = \max_i |t_{i+1} - t_i|$.

Proof. Given $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(s) - f(t)| \le \epsilon$ whenever $|s - t| \le \delta$. (The function f is also uniformly continuous on $[0, 1]$.) For any grid $\mathbb{G}$ with $\text{mesh}(\mathbb{G}) \le \delta$ we have
$$\mathcal{Q}(f, \mathbb{G}) \le \sum_i \epsilon |f(t_{i+1}) - f(t_i)| \le \epsilon \mathcal{V}(f)$$

□ And so on.

Let $\{X_t : 0 \le t \le 1\}$ be the initial chunk of a standard Brownian motion.

- Fact: Almost almost all sample paths of X have infinite total variation.

Proof. Write $V(\omega)$ for the total variation of the sample path $X(\cdot, \omega)$. For each positive integer n, define

$$V_n(\omega) = \sum\nolimits_i |X(t_{i+1}, \omega) - X(t_i, \omega)| \qquad \text{where } t_i = i/n \text{ for } i = 0, 1, \dots, n.$$

The absolute value of the increments $\Delta_i X = X(t_{i+1}, \omega) - X(t_i, \omega)$ are independent with mean $c_0/\sqrt{n}$ and variance less than c_1/n, for some constants c_0 and c_1. Thus

$$\mathbb{E}V_n = c_0\sqrt{n} \qquad \text{and} \qquad \text{var}(V_n) \le c_1.$$

By Tchebychev's inequality,

$$\mathbb{P}\{V_n \ge \tfrac{1}{2}c_0\sqrt{n}\} = \mathbb{P}\{V_n - c_0\sqrt{n} \ge -\tfrac{1}{2}c_0\sqrt{n}\} \ge 1 - \frac{\text{var}(V_n)}{(\frac{1}{2}c_0\sqrt{n})^2} \to 1 \qquad \text{as } n \to \infty.$$

□ Complete the proof by noting that $V(\omega) \ge V_n(\omega)$ for every n.

- Fact: For any sequence of grids $\mathbb{G}_n$ with $\text{mesh}(\mathbb{G}_n) \to 0$,

$$\mathcal{Q}(X(\cdot, \omega), \mathbb{G}_n) \to 1 \qquad \text{in probability}$$

Proof. Abbreviate $\mathcal{Q}(X(\cdot, \omega), \mathbb{G}_n)$ to $\mathcal{Q}_n(\omega)$. That is,

$$\mathcal{Q}_n(\omega) = \sum\nolimits_i (\Delta_i X)^2 \qquad \text{where } \delta_i X \text{ are the increments in } X \text{ for grid } \mathbb{G}_n.$$

Notice that

$$\mathbb{E}\mathcal{Q}_n = \sum\nolimits_i \mathbb{E}(\Delta_i X)^2 = \sum\nolimits_i \left(t_{i+1} - t_i\right) = 1$$

and, by the independence of the increments,

$$\text{var}(\mathcal{Q}_n) = \sum\nolimits_i \text{var}\left((\Delta_i X)^2\right) \le \sum\nolimits_i \mathbb{E}(\Delta_i X)^4 \le c_2 \sum\nolimits_i \left(t_{i+1} - t_i\right)^2 \le c_2 \text{mesh}(\mathbb{G}_n) \to 0,$$

□ for some constant c_2.