Written very late at night. Not yet checked. Very tired. Brain imploding. Do not invest money based on the calculations in this handout.

GEOMETRIC BROWNIAN MOTION AND THE BLACK-SCHOLES MODEL

From the handout on the Itô formula, you know that

GBM <1> $$S_t = \exp\left(\sigma B_t + \alpha t\right) \qquad \text{with } B \text{ a Brownian motion}$$

then

$$S_t = 1 + \sigma S \bullet B_t + (a + \tfrac{1}{2}\sigma^2) S \bullet \mathcal{U}_t \qquad \text{where } \mathcal{U}_t \equiv t.$$

In particular if we put $\alpha = \mu - \frac{1}{2}\sigma^2$, for a constant μ, then

$$S_t = 1 + \sigma S \bullet B_t + \mu S \bullet \mathcal{U}_t,$$

or, in more traditional notation,

$$dS_t = \sigma S_t dB_t + \mu S_t dt \qquad \text{or} \qquad \frac{dS_t}{S_t} = \sigma dB_t + \mu dt.$$

Over a small time interval, $[t, t+\delta]$ the proportional change $\Delta S/S_t$ in S is approximately $N(\mu\delta, \sigma^2\delta)$ distributed.

In the special case where μ is zero, $S_t = 1 + \sigma S \bullet B_t$, which is a martingale.

The process S from <1> with $\alpha = \mu - \frac{1}{2}\sigma^2$ is called a ***geometric Brownian motion***. It is often used to model a stock price over time.

Notice that I have implicitly standardized the price so that $S_0 = 1$. In effect, S_t measures the price relative to the initial price. I will also make another standardization by assuming that the interest rate is zero, so that I don't have to discount future returns or introduce a bond into the calculations.

[§GHX] 1. Stochastic integrals with respect to stochastic integrals

To apply the Itô formula in the arbitrage argument in the next section I will need a little piece of the calculus for stochastic integrals, namely

GHX <2> $$G \bullet (H \bullet X) = (GH) \bullet X.$$

Let me prove the equality only for elementary processes

$$G(t,\omega) = \sum\nolimits_{i=0}^{n} g_i(\omega)\mathbb{I}\{t_i < t \le t_{i+1}\}$$
$$H(t,\omega) = \sum\nolimits_{i=0}^{n} h_i(\omega)\mathbb{I}\{t_i < t \le t_{i+1}\}.$$

I postpone to a more rigorous course the formal passage to the limit from the elementary case to a case general enough to handle the process in the next section.

There is no loss of generality in assuming that both G and H are step functions for the same grid (we could always work with a common refinement of the grid for G and the grid for H) and that we wish to establish equality <2> at a grid point (we could always add extra points to the grid). Writing Z for the process $H \bullet X$, we have $Z_{t_k} = \sum_{i<k} h_i \Delta_i X$, so that

$$\Delta_k Z = Z_{t_{k+1}} - Z_{t_k} = h_k \Delta_k X,$$

and

$$G \bullet Z_{t_k} = \sum\nolimits_{i<k} g_i \Delta_i Z = \sum\nolimits_{i<k} g_i h_i \Delta_i X = (GH) \bullet X_{t_k},$$

as asserted.

[§BSpde]

2. The Black-Scholes partial differential equation

Suppose we are trying to price an option that returns an amount $\psi(S_1)$ at time $t = 1$, where ψ is a known function, such as $\psi(x) = (x - K)^+$ with K constant. We might hope that the price at time t would be a function $f(S_t, t)$ of the stock price and time. That is, we could try to find a function f such that the payment of $\mathcal{P}_t = f(S_t, t)$ at time t for a return $\psi(S_1)$ at time $t = 1$ presents no arbitrage opportunities.

We must have $\mathcal{P}_1 = \psi(S_1)$, for therwise there is an obvious arbitrage involving buying and selling at time $t = 1$.

Assuming suitable smoothess for f, we get via the Itô formula a stochastic integral representation for the price,

price.pde <3>
$$\mathcal{P}_t = f(1, 0) + F_x \bullet S_t + \tfrac{1}{2} F_{xx} \bullet A_t + F_y \bullet \mathcal{U}_t,$$

where

$$F_x(t) = f_x(S_t, t) \qquad \text{and} \qquad F_{xx}(t) = f_{xx}(S_t, t) \qquad \text{and} \qquad F_y(t) = f_y(S_t, t)$$

and A_t is the compensator for $\sigma S \bullet B_t$, the martingale part of S_t. We know B has compensator $\mathcal{U}$. And from Lemma 16 on the stochastic integral handout, we know that $\sigma S \bullet B$ has compensator $A_t = (\sigma^2 S^2) \bullet \mathcal{U}$. Thus

$$F_{xx} \bullet A_t = F_{xx} \bullet \left((\sigma^2 S^2) \bullet \mathcal{U}\right) = \sigma^2 (F_{xx} S^2) \bullet \mathcal{U}_t \qquad \text{by } <2>.$$

If we denote the as-yet-unknown constant $f(1, 0)$ by c_0 then <3> becomes

$$\mathcal{P}_t - c_0 = F_x \bullet S_t + \left(\tfrac{1}{2}\sigma^2 F_{xx} + F_y\right) \bullet \mathcal{U}_t$$

The final term vanishes if f satisfies the partial differential equation

$$\tfrac{1}{2}\sigma^2 x^2 f_{xx}(x, y) + f_y(x, y) = 0$$

If f also satisfies the boundary condition $f(x, 1) = \psi(x)$ then

$$\mathcal{P}_t - c_0 = F_x \bullet S_t \qquad \text{and} \qquad \mathcal{P}_1 = \psi(S_1).$$

Interpret $F_x \bullet S_t$ as the profit from the trading strategy F_x up to time t. The last equation then says that we can arrange to make $\psi(S_1) - c_0$ by a trading scheme. If we could obtain $\psi(S_1)$ by paying any price but c_0 at time $t = 0$ then we could make a riskless profit.

pricet <4> **Exercise.** Show that $\mathcal{P}_t$ is the price to pay at time t if there is to be no arbitrage opportunity. □

You can learn how to solve the PDE with boundary condition $f(x, 1) = \psi(x)$ by consulting a book such as Wilmott, Howison & Dewynne (1995, Chapter 5).

[§girsanov]

3. Change of measure

If you don't like solving PDEs you might prefer a sneakier way to get at the solution, but it depends on some heavier machinery from stochastic calculus.

BMsi <5> **Fact.** *Suppose Z is a random variable that might depend on the whole Brownian motion sample path $B(\cdot, \omega)$ (in a suitably measurable way). If $\mathbb{E}Z^2 < \infty$ (or even under some slightly weaker assumptions), there exists a constant C and an (adapted, ...) process H such that $Z = C + H \bullet B_1$.*

REMARK. If you are prepared to tackle some rigorous probability, you will find a fairly self-contained proof in Pollard (2001, Section 9.7). The proof could even be simplified a bit by an appeal to the the Itô formula.

Let me creep up on the main argument with tiny steps.

First you need to know what a change of measure is. If $q(\omega)$ is a nonnegtive random variable, living on a set Ω already equipped with a probability, define

$$\mathbb{Q}A = \mathbb{E}_{\mathbb{P}}\left(q(\omega)\mathbb{I}\{\omega \in A\}\right) \qquad \text{for } A \subseteq \Omega.$$

The subscript $\mathbb{P}$ on the expectation will be needed to avoid confusion when we have more than one probability defined on Ω. Provided $\mathbb{E}_{\mathbb{P}} q = 1$, the new $\mathbb{Q}$ is a genuine probability. It satisfies all the usual properties, such as

$$\mathbb{Q}\left(\cup_{n\in\mathbb{N}} A_n\right) = \sum\nolimits_{n\in\mathbb{N}} \mathbb{Q}(A_n) \qquad \text{for disjoint events } \{A_n\}$$

and $\mathbb{Q}\emptyset = 0$ and $\mathbb{Q}\Omega = 1$. More useful is the formula

$$\mathbb{E}_{\mathbb{Q}} X = \mathbb{E}_P(Xq) \qquad \text{for a random variable } X.$$

The random variable q is often called the density of $\mathbb{Q}$ with respect to $\mathbb{P}$.

normalCoM <6> **Example.** Suppose $X_1, \ldots, X_k$ are independent random variables with $X_i \sim N(0, \sigma_i^2)$, under the probability distribution $\mathbb{P}$. For arbitrary constants $\{\alpha_i\}$ define a new probability, $\mathbb{Q}$, by means of the density

$$q = \exp\left(\sum\nolimits_i (\alpha_i X_i - \tfrac{1}{2}\alpha_i^2\sigma_i^2)\right).$$

Recall the formula $\exp(\theta\mu + \frac{1}{2}\sigma^2\theta^2)$ for the moment generating function $\mathbb{E}\exp(\theta Z)$ for a random variable Z with a $N(\mu, \sigma^2)$ distribution. Together with the independence of the X_i's under $\mathbb{P}$, this formula ensures that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}} q &= \exp\left(-\sum\nolimits_i \alpha_i^2\sigma_i^2\right)\prod\nolimits_i \mathbb{E}_{\mathbb{P}}\exp(\alpha_i X_i) \\ &= \exp\left(-\sum\nolimits_i \alpha_i^2\sigma_i^2\right)\prod\nolimits_i \exp(\tfrac{1}{2}\alpha_i^2\sigma^2) = 1.\end{aligned}$$

The $\mathbb{Q}$ is a genuine probability distribution.

We can calculate the joint moment generating function of $X_1, \ldots, X_k$ under $\mathbb{Q}$ by a similar argument. For constants $\theta_1, \ldots, \theta_k$,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\exp\left(\sum\nolimits_i \theta_i X_i\right) &= \mathbb{E}_{\mathbb{P}}\exp\left(\sum\nolimits_i \theta_i X_i + \sum\nolimits_i (\alpha_i X_i - \tfrac{1}{2}\alpha_i^2\sigma_i^2)\right) \\ &= \prod\nolimits_i \mathbb{E}_{\mathbb{P}}\exp\left((\theta_i + \alpha_i)X_i - \tfrac{1}{2}\alpha_i^2\sigma_i^2\right) \\ &= \prod\nolimits_i \exp\left(\tfrac{1}{2}(\theta_i + \alpha_i)^2\sigma_i^2 - \tfrac{1}{2}\alpha_i^2\sigma_i^2\right) \\ &= \prod\nolimits_i \exp\left(\tfrac{1}{2}\theta_i^2\sigma_i^2 + \theta_i\alpha_i\sigma_i^2\right)\end{aligned}$$

The last expression is a product of moment generating functions for $N(\alpha_i\sigma_i^2, \sigma_i^2)$ distributions. Under $\mathbb{Q}$ the random variables $X_1, \ldots, X_k$ are still independent normals, with the same variances as under $\mathbb{P}$, but now the means have changed: $\mathbb{E}_{\mathbb{Q}} X_i = \alpha_i\sigma_i^2$. □

The change-of-measure trick also works for infinite collections of normally distributed random variables.

BMCoM <7> **Example.** Suppose $\{B_t : 0 \le t \le 1\}$ is a standard Brownian motion under $\mathbb{P}$. For a fixed constant α, define

$$q(\omega) = \exp\left(\alpha B_1(\omega) - \tfrac{1}{2}\alpha^2\right)$$

Note that $\mathbb{E}_{\mathbb{P}} q = 1$ because $B_1 \sim N(0, 1)$ under $\mathbb{P}$.

If we change the measure to the $\mathbb{Q}$ defined by the density q with respect to $\mathbb{P}$, we do not change the continuity of the sample paths of B. Suppose $0 = t_0 < t_1 < \ldots < t_{n+1} = 1$ is a grid with corresponding increments $\Delta_i B$ for B. Under $\mathbb{P}$ the increments are independent with $\Delta_i B \sim N(0, \delta_i)$, where $\delta_i = t_{i+1} - t_i$. The density q can also be written as

$$q = \exp\left(\sum\nolimits_{i=0}^n (\alpha\Delta_i B - \tfrac{1}{2}\alpha^2\delta_i)\right).$$

From Example <6> with $\sigma_i^2 = \delta_i$, deduce that the increments are again independent under $\mathbb{Q}$ with $\Delta_i B \sim N(\alpha\delta_i, \delta_i)$, or $\Delta_i B - \alpha\delta_i \sim N(0, \delta_i)$. The process $\tilde{B}_t = B_t - \alpha t$ has all the properties needed to characterize it as a Brownian motion under $\mathbb{Q}$. □

Now reconsider the stock prices modelled as a geometric Brownian motion,

$$S_t = \exp\big(\sigma B_t + (\mu - \tfrac{1}{2}\sigma^2)t\big) \qquad \text{with } B \text{ a Brownian motion under } \mathbb{P}.$$

If $\mathbb{Q}$ has density

$$q = \exp\big(-\mu B_1 - \tfrac{1}{2}\mu^2\big)$$

with respect to $\mathbb{P}$ then $\tilde{B}_t = B_t + \mu t$ is Brownian motion under $\mathbb{Q}$ and

$$S_t = \exp\Big(\sigma\tilde{B}_t - \tfrac{1}{2}\sigma^2 t\Big) \qquad \text{with } \tilde{B} \text{ a Brownian motion under } \mathbb{Q}.$$

With the change of measure we have effectively eliminated the drift coefficient μ. Under $\mathbb{Q}$, the stock price is a martingale driven by the Brownian motion $\tilde{B}$.

Once again consider an option that promises to deliver a random amount Z at time $t = 1$. The variable could depend of the stock price history in a complicated way. For example, we could contemplate a most exotic option that delivers

$$Z = \max_{0\le t\le 1} S_t - \sum_{j=107}^{233} S_{100/j}^2 \sin(S_{j/1000}) + \int_0^1 \cos(S_t^3)\,dt$$

at time $t = 1$. What matters most is that Z can also be thought of as a (weird) function of the $\tilde{B}$ sample path: just insert $\exp(\sigma\tilde{B}_t - \frac{1}{2}\sigma^2 t)$ wherever you see an S_t in the definition of Z, for various t. The function also depends on σ, but there is no μ in sight.

The dramatic moment arrives.

Appeal to Fact <5> for the Brownian motion $\tilde{B}$ to express Z as

$$Z = C + H \bullet \tilde{B}_1$$

for some constant C and some adapted process H. (Maybe you should check that $\mathbb{E}_{\mathbb{Q}} Z^2 < \infty$ for the Z you have in mind.) If we could trade directly in $\tilde{B}$, we could interpret $H \bullet \tilde{B}$ as a trading scheme. We need to convert to a scheme trading in the stock price by means of the representation

$$S_t = 1 + \sigma S \bullet \tilde{B}_t \qquad \text{under } \mathbb{Q}.$$

The equality <2> again comes to the rescue, if we integrate the process $1/S_t$ with respect to the processes on both sides of the previous display.

$$\begin{aligned}(1/S) \bullet S_t &= (1/S) \bullet 1_t + (1/S) \bullet (\sigma S \bullet \tilde{B})_t \\ &= 0 + \sigma(S/S) \bullet \tilde{B}_t \qquad \text{cf. increments of a constant process} \\ &= \sigma\tilde{B}_t.\end{aligned}$$

Similarly,

$$H \bullet \tilde{B}_t = \frac{1}{\sigma} H \bullet \big((1/S) \bullet S\big)_t = \frac{1}{\sigma}(H/S) \bullet S_t.$$

Write K_t for $(1/\sigma)(H/S)_t$. Then we have a trading scheme to recover the amount $Z - C$ at time $t = 1$:

$$Z = C + K \bullet S_1$$

You might be a bit disappointed that you know only how to trade under $\mathbb{Q}$ if in fact you live in the world where $\mathbb{P}$ is in control and S is not a martingale because of that pesky, unknown μ. (You did say that you knew the value of σ, didn't you?)

Not to worry. Think of K as a shorthand for a sequence of elementary processes,

$$K_n(t) = \sum\nolimits_j k_{n,j}(\omega)\mathbb{I}\{t_{n,j} < t \le t_{n,j+1}\}$$

for which $K_n \bullet S$ converges to $K \bullet S$. The trading scheme K_n can be spelled out as

for each j: buy $k_{n,j}$ shares at time $t_{n,j}$ then sell them at time $t_{n,j+1}$

At this point I need to be a little more precise about the sense of the convergence. In fact, I need (and the stochastic calculus gives) convergence in $\mathbb{Q}$ probability:

$$\mathbb{Q}\{|K_n \bullet S - K \bullet S| > \epsilon\} \to 0 \qquad \text{as } n \to \infty, \text{ for each } \epsilon > 0.$$

The nice thing about the relationship between $\mathbb{Q}$ and $\mathbb{P}$ is: sequences that converge in $\mathbb{Q}$-probability also converge (to the same thing) in $\mathbb{P}$-probability. The idealized trading scheme K is a limit of the elementary schemes K_n under both $\mathbb{Q}$ and $\mathbb{P}$.

Some calculations needed here.

Before we leave the $\mathbb{Q}$-world, note that S and $K \bullet S$ are both martingales under $\mathbb{Q}$. In particular,

$$0 = \mathbb{E}_{\mathbb{Q}} K \bullet S_0 = \mathbb{E}_{\mathbb{Q}} K \bullet S_1$$

and hence

$$\mathbb{E}_{\mathbb{Q}} Z = C,$$

a calculation that we could, in principle, carry out.

Back in the world controlled by $\mathbb{P}$, we therefore have a trading scheme, K, that delivers the amount $Z - C$ at time $t = 1$. We should pay C at time $t = 0$ to receive Z at time $t = 1$.

In short: to find the price to pay at time $t = 0$ for receiving Z at time $t = 1$,

(i) Find the probability measure $\mathbb{Q}$ that makes S a $\mathbb{Q}$-martingale.

(ii) Hope (or invoke some probability theorem to show) that convergence in $\mathbb{Q}$-probability is the same as convergence in $\mathbb{P}$-probability.

(iii) Calculate the price as $C = \mathbb{E}_{\mathbb{Q}} Z$.

BScall <8> **Example.** Suppose $Z = (S_1 - K)^+$, which I believe is the return from the option known as a call with strike price K. Calculate.

$$C = \mathbb{E}_{\mathbb{Q}} \exp(\sigma S_1 - K)^+ = \mathbb{E}_{\mathbb{Q}} \Big(\exp(\sigma \tilde{B}_1 - \tfrac{1}{2}\sigma^2) - K \Big)^+ .$$

Under $\mathbb{Q}$, the random variable $W = \tilde{B}_1$ has a standard normal distribution. Also

$$\exp(\sigma W - \tfrac{1}{2}\sigma^2) \geq K \text{ if and only if } W \geq L := \frac{1}{\sigma} \log K + \tfrac{1}{2}\sigma$$

Write $\overline{\Phi}(t) = 1 - \Phi(t)$ for the standard normal tail probability. Calculate.

$$\begin{aligned} C &= \mathbb{E}_{\mathbb{Q}} \big(\exp(\sigma W - \tfrac{1}{2}\sigma^2) - K \big) \mathbb{I}\{W \geq L\} \\ &= \frac{1}{\sqrt{2\pi}} \int_L^\infty \exp(\sigma x - \tfrac{1}{2}\sigma^2 - \tfrac{1}{2}x^2)\, dx - K \mathbb{E}_{\mathbb{Q}} \mathbb{I}\{W \geq L\} \\ &= \frac{1}{\sqrt{2\pi}} \int_L^\infty \exp(-\tfrac{1}{2}(x - \sigma)^2)\, dx - K \overline{\Phi}(L) \\ &= \overline{\Phi}(L - \sigma) - K \overline{\Phi}(L) \end{aligned}$$

I sure hope the last expression agrees with the textbooks for the case where $S_0 = 1$ and there is a zero interest rate. Stay tuned for the corrected version with the correct result. □

References

Pollard, D. (2001), *A User's Guide to Measure Theoretic Probability*, Cambridge University Press.

Wilmott, P., Howison, S. & Dewynne, J. (1995), *The Mathematics of Financial Derivatives: a Student Introduction*, Cambridge University Press.