Example 4: Coin tossing game: HHH vs. TTHH

Here is a coin tossing game that illustrates how conditioning can break a complex random mechanism into a sequence of simpler stages. Imagine that I have a fair coin, which I toss repeatedly. Two players, M and R, observe the sequence of tosses, each waiting for a particular pattern on consecutive tosses.

M waits for hhh
R waits for tthh.

The one whose pattern appears first is the winner. What is the probability that M wins?

For example, the sequence ththhttthh . . . would result in a win for R, but ththhthhh . . . would result in a win for M.

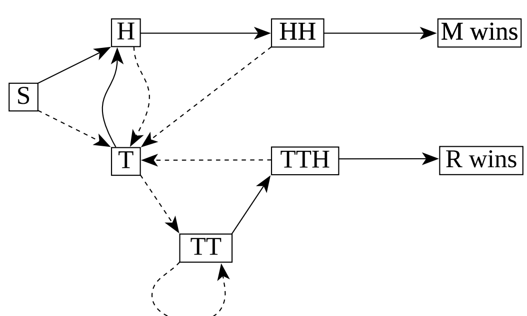
At first thought one might imagine that M has the advantage. After all, surely it must be easier to get a pattern of length 3 than a pattern of length 4. You'll discover that the solution is not that straightforward.

The possible states of the game can be summarized by recording how much of his pattern each player has observed (ignoring false starts, such as hht for M, which would leave him back where he started, although R would have matched the first t of his pattern.).

States	**M partial pattern**	**R partial pattern**
$\boxed{\text{S}}$	–	–
$\boxed{\text{H}}$	h	–
$\boxed{\text{T}}$	–	t
$\boxed{\text{TT}}$	–	tt
$\boxed{\text{HH}}$	hh	–
$\boxed{\text{TTH}}$	h	tth
$\boxed{\text{M wins}}$	hhh	?
$\boxed{\text{R wins}}$	?	tthh

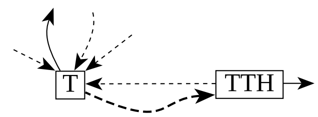
By claiming that these states summarize the game I am tacitly assuming that the coin has no "memory", in the sense that the conditional probability of a head given any particular past sequence of heads and tails is 1/2 (for a fair coin). The past history leading to a particular state does not matter; the future evolution of the game depends only on what remains for each player to achieve his desired pattern.

The game is nicely summarized by a diagram with states represented by little boxes joined by arrows that indicate the probabilities of transition from one state to another. Only transitions with a nonzero probability are drawn. In this problem each nonzero probability equals 1/2. The solid arrows correspond to transitions resulting from a head, the dotted arrows to a tail.



For example, the arrows leading from $\boxed{\text{S}}$ to $\boxed{\text{H}}$ to $\boxed{\text{HH}}$ to $\boxed{\text{M wins}}$ correspond to heads; the game would progress in exactly that way if the first three tosses gave hhh. Similarly the arrows from $\boxed{\text{S}}$ to $\boxed{\text{T}}$ to $\boxed{\text{TT}}$ correspond to tails.

The arrow looping from $\boxed{\text{TT}}$ back into itself corresponds to the situation where, after ...tt, both players progress no further until the next head. Once the game progresses down the arrow $\boxed{\text{T}}$ to $\boxed{\text{TT}}$ the step into $\boxed{\text{TTH}}$ becomes inevitable. Indeed, for the purpose of calculating the probability that M wins, we could replace the side branch by:



The new arrow from $\boxed{\text{T}}$ to $\boxed{\text{TTH}}$ would correspond to a sequence of tails followed by a head. With the state $\boxed{\text{TT}}$ removed, the diagram would become almost symmetric with respect to M and R. The arrow from $\boxed{\text{HH}}$ back to $\boxed{\text{T}}$ would show that R actually has an advantage: the first h in the tthh pattern presents no obstacle to him.

Once we have the diagram we can forget about the underlying game. The problem becomes one of following the path of a particle that moves between the states according to the transition probabilities on the arrows. The original game has $\boxed{\text{S}}$ as its starting state, but it is just as easy to solve the problem for a particle starting from any of the states. The method that I will present actually solves the problems for all possible starting states by setting up equations that relate the solutions to each other. Define probabilities for the particle:

$$\begin{aligned} P_S &= \mathbb{P}\{\text{reach } \boxed{\text{M wins}} \mid \text{start at } \boxed{\text{S}}\} \\ P_T &= \mathbb{P}\{\text{reach } \boxed{\text{M wins}} \mid \text{start at } \boxed{\text{T}}\} \end{aligned}$$

and so on. I'll still refer to the solid arrows as "heads", just to distinguish between the two arrows leading out of a state, even though the coin tossing interpretation has now become irrelevant.

Calculate the probability of reaching $\boxed{\text{M wins}}$, under each of the different starting circumstances, by breaking according to the result of the first move, and then conditioning.

$$\begin{aligned} P_S &= \mathbb{P}\{\text{reach } \boxed{\text{M wins}}, \text{heads} \mid \text{start at } \boxed{\text{S}}\} + \mathbb{P}\{\text{reach } \boxed{\text{M wins}}, \text{tails} \mid \text{start at } \boxed{\text{S}}\} \\ &= \mathbb{P}\{\text{heads} \mid \text{start at } \boxed{\text{S}}\}\mathbb{P}\{\text{reach } \boxed{\text{M wins}} \mid \text{start at } \boxed{\text{S}}, \text{heads}\} \\ &\quad + \mathbb{P}\{\text{tails} \mid \text{start at } \boxed{\text{S}}\}\mathbb{P}\{\text{reach } \boxed{\text{M wins}} \mid \text{start at } \boxed{\text{S}}, \text{tails}\}. \end{aligned}$$

The lack of memory in the fair coin reduces the last expression to $\frac{1}{2}P_H + \frac{1}{2}P_T$. Notice how "start at $\boxed{\text{S}}$, heads" has been turned into "start at $\boxed{\text{H}}$" and so on. We have our first equation:

$$P_S = \tfrac{1}{2}P_H + \tfrac{1}{2}P_T.$$

Similar splitting and conditioning arguments for each of the other starting states give

$$\begin{aligned} P_H &= \tfrac{1}{2}P_T + \tfrac{1}{2}P_{HH} \\ P_{HH} &= \tfrac{1}{2} + \tfrac{1}{2}P_T \\ P_T &= \tfrac{1}{2}P_H + \tfrac{1}{2}P_{TT} \\ P_{TT} &= \tfrac{1}{2}P_{TT} + \tfrac{1}{2}P_{TTH} \\ P_{TTH} &= \tfrac{1}{2}P_T + 0. \end{aligned}$$

We could use the fourth equation to substitute for P_{TT}, leaving

$$P_T = \tfrac{1}{2}P_H + \tfrac{1}{2}P_{TTH}.$$

This simple elimination of the P_{TT} contribution corresponds to the excision of the $\boxed{\text{TT}}$ state from the diagram. If we hadn't noticed the possibility for excision the algebra would have effectively done it for us. The six splitting/conditioning arguments give six linear equations in six unknowns. If you solve them you should get $P_S = 5/12$, $P_H = 1/2$, $P_T = 1/3$, $P_{HH} = 2/3$, and $P_{TTH} = 1/6$. For the original problem, M has probability 5/12 of winning. □

There is a more systematic way to carry out the analysis in the last problem without drawing the diagram. The transition probabilities can be installed into an 8 by 8 matrix whose rows and columns are labeled by the states:

$$P = \begin{array}{c} \\ \boxed{\text{S}} \\ \boxed{\text{H}} \\ \boxed{\text{T}} \\ \boxed{\text{HH}} \\ \boxed{\text{TT}} \\ \boxed{\text{TTH}} \\ \boxed{\text{M wins}} \\ \boxed{\text{R wins}} \end{array} \begin{array}{c} \begin{array}{cccccccc} \boxed{\text{S}} & \boxed{\text{H}} & \boxed{\text{T}} & \boxed{\text{HH}} & \boxed{\text{TT}} & \boxed{\text{TTH}} & \boxed{\text{M wins}} & \boxed{\text{R wins}} \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

If we similarly define a column vector,

$$\pi = (P_S, P_H, P_T, P_{HH}, P_{TT}, P_{TTH}, P_{\text{M wins}}, P_{\text{R wins}})',$$

then the equations that we needed to solve could be written as

$$P\pi = \pi,$$

with the boundary conditions $P_{\text{M wins}} = 1$ and $P_{\text{R wins}} = 0$. I didn't bother adding the equations $P_{\text{M wins}} = 1$ and $P_{\text{R wins}} = 0$ to the list of equations; they correspond to the isolated terms 1/2 and 0 on the right-hand sides of the equations for P_{HH} and P_{TTH}.

The matrix P is called the **transition matrix**. The element in row i and column j gives the probability of a transition from state i to state j. For example, the third row, which is labeled $\boxed{\text{T}}$, gives transition probabilities from state $\boxed{\text{T}}$. If we multiply P by itself we get the matrix P^2, which gives the "two-step" transition probabilities. For example, the element of P^2 in row $\boxed{\text{T}}$ and column $\boxed{\text{TTH}}$ is given by

$$\sum_j P_{T,j} P_{j,TTH} = \sum_j \mathbb{P}\{\text{step to j} \mid \text{start at } \boxed{\text{T}}\}\mathbb{P}\{\text{step to } \boxed{\text{TTH}} \mid \text{start at j}\}.$$

Here j runs over all states, but only $j = \boxed{\text{H}}$ and $j = \boxed{\text{TT}}$ contribute nonzero terms. Substituting

$$\mathbb{P}\{\text{reach } \boxed{\text{TTH}} \text{ in two steps} \mid \text{start at } \boxed{\text{T}}, \text{ step to j}\}$$

for the second factor in the sum, we get the splitting/conditioning decomposition for

$$\mathbb{P}\{\text{reach } \boxed{\text{TTH}} \text{ in two steps} \mid \text{start at } \boxed{\text{T}}\},$$

a two-step transition possibility.

Questions: What do the elements of the matrix P^n represent? What happens to this matrix as n tends to infinity? See the output from the MatLab m-file Markov.m.

The name **Markov chain** is given to any process representable as the movement of a particle between states (boxes) according to transition probabilities attached to arrows connecting the various states. The sum of the probabilities for arrows leaving a state should add to one. All the past history except for identification of the current state is regarded as irrelevant to the next transition; given the current state, the past is conditionally independent of the future.