Lévy's martingale characterization of Brownian motion

Suppose $\{X_t : 0 \le t \le 1\}$ a martingale with continuous sample paths and $X_0 = 0$. Suppose also that $X_t^2 - t$ is a martingale. Then X is a Brownian motion.

Heuristics. I'll give a rough proof for why X_1 is $N(0,1)$ distributed.

First note that the two martingale assuptions give two properties of the increment $\Delta X = X_t - X_s$, for $s < t$. Write $\mathbb{E}_s(\ldots)$ for expectations conditional on the information, $\mathcal{F}_s$, up to time s. The martingale properties are

$$\mathbb{E}_s\left(X_s + \Delta X\right) = X_s$$
$$\mathbb{E}_s\left(X_s^2 + 2(\Delta X)X_s + (\Delta X)^2 - t\right) = X_s^2 - s$$

Using the fact that X_s can be treated like a constant when conditioning on $\mathcal{F}_s$, we have

$$\mathbb{E}_s \Delta X = 0$$

and

$$\mathbb{E}_s(\Delta X)^2 = t - s - 2X_s(\mathbb{E}_s \Delta X) = t - s.$$

Put another way, for random variables W_1 and W_2 that depend only on information up to time s,

<1> $$\mathbb{E}\left(W_1 \Delta X\right) = 0$$

<2> $$\mathbb{E}\left(W_2 (\Delta X)^2\right) = (t-s)\mathbb{E}W_2$$

Let $f(x,t)$ be a smooth function of two arguments, $x \in \mathbb{R}$ and $t \in [0,1]$. Define

$$f_x = \frac{\partial f}{\partial x} \qquad \text{and} \qquad f_{xx} = \frac{\partial^2 f}{\partial^2 x} \qquad \text{and} \qquad f_t = \frac{\partial f}{\partial t}.$$

Let $h = 1/n$ for some large positive integer n. Define $t_i = ih$ for $i = 0, 1, \ldots, n$. Write $\Delta_i X$ for $X(t_i + h) - X(t_i)$. Then

$$\begin{aligned}\mathbb{E}f(X_1,1) - \mathbb{E}f(X_0,0) &= \sum_{i<n}\left(\mathbb{E}f(X_{t_i+h}, t_i+h) - \mathbb{E}f(X_{t_i}, t_i)\right) \\ &\approx \sum_{i<n}\mathbb{E}\left((\Delta_i X) f_x(X_{t_i}, t_i) + \tfrac{1}{2}(\Delta_i X)^2 f_{xx}(X_{t_i}, t_i) + h f_t(X_{t_i}, t_i)\right)\end{aligned}$$

For the ith sumand, invoke <1> with $W_1 = f_x(X_{t_i}, t_i)$ and <2> with $W_2 = f_{xx}(X_{t_i}, t_i)$, for $s = t_i$ and $t = t_{i+1} = s + h$. The summand simplifies to

$$\tfrac{1}{2}\mathbb{E}(\Delta_i X)^2 \mathbb{E} f_{xx}(X_{t_i}, t_i) + h\mathbb{E} f_t(X_{t_i}, t_i)$$

The sum over the grid then takes the form of an approxiating sum for the integral

$$\int_0^1 \left(\tfrac{1}{2}\mathbb{E}f_{xx}(X_s, s) + \mathbb{E}f_t(X_s, s)\right) ds$$

If we paid more attention to the errors of approximation we would see that their contributions go to zero as the $\{t_i\}$ grid gets finer. In the limit we have

$$\mathbb{E}f(X_1,1) - \mathbb{E}f(X_0,0) = \mathbb{E}\int_0^t \left(\tfrac{1}{2} f_{xx}(X_s, s) + f_t(X_s, s)\right) ds$$

Now specialize to the case $f(x,s) = \exp\left(\theta x - \frac{1}{2}\theta^2 s\right)$, with θ a fixed constant. By direct calculation,

$$f_x = \theta f(x,s) \qquad \text{and} \qquad f_{xx} = \theta^2 f(x,s) \qquad \text{and} \qquad f_t = -\tfrac{1}{2}\theta^2 f(x,s)$$

Thus

$$\mathbb{E}e^{\theta X_1} e^{-\theta^2/2} - 1 = \int_0^1 0\, ds = 0.$$

That is, X_1 has the moment generating function $\exp(\theta^2/2)$, which identifies it as having a $N(0,1)$ distribution.

