Statistics 251/551b **Due: Wednesday 21 January**
2004: Sheet 1

The final grade for each person in the class will be based on the problems marked with a ✠ symbol plus the project. The other problems will tend to be harder. They will be counted as bonus questions. The final grade will be adjusted upward for any student who accumulates a goodly supply of bonus points.

✠ (1.1) Please print clearly your name, email address, major (or graduate department) and year.

✠ (1.2) Reconsider the problem where a stack of two coins is "shuffled" by inverting either the top coin (with probability θ_1) or the whole stack (with probability θ_2), as in the first lecture. Assume $\theta_1 > 0$ and $\theta_2 > 0$. Suppose both coins initially have head facing up. Let H_n denote the number of heads facing up after n shuffles.

(i) [5 points] Find $\mathbb{P}\{H_3 = 2 \mid H_1 = 1, H_2 = 1\}$.

(ii) [5 points] Find $\mathbb{P}\{H_3 = 2 \mid H_2 = 1\}$. Deduce that $\{H_n : n = 0, 1, 2, \dots\}$ is not a Markov chain.

✠ (1.3) [5 points] Consider a Markov chain on a state space $\mathcal{S} = \{1, 2, 3, 4, 5\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{pmatrix}$$

Find $\{n \in \mathbb{N} : \mathbb{P}_1\{X_n = 1\} > 0\}$, without appealing to any theorems about periodicity. [It would help if you explained some of your reasoning and did not just give a list of numbers.]

✠ (1.4) [10 points] Chang Exercise 1.8.

(1.5) Let $\{X_n : n \in \mathbb{N}\}$ be a Markov chain with a countably infinite state space $\mathcal{S}$. Suppose there is an initial distribution μ for which $\mathbb{P}_\mu\{X_n = j\} \to \pi_j$ as $n \to \infty$, for each $j \in \mathcal{S}$.

(i) Prove (rigorously) that $\pi_j \ge \sum_{i \in \mathcal{S}} \pi_i P(i, j)$, for each j.

(ii) Use the fact that $\sum_{j \in \mathcal{S}} P(i, j) = 1$ for each i to conclude that $\pi_j = \sum_{i \in \mathcal{S}} \pi_i P(i, j)$, for each j.