Statistics 251/551 **Due: Wednesday 4 February**
2004: Sheet 2

Remember: The final grade for each person in the class will be based on the problems marked with a ✠ symbol plus the project. The other problems will tend to be harder. They will be counted as bonus questions. The final grade will be adjusted upward for any student who accumulates a goodly supply of bonus points.

For the first three problems, suppose i and j are recurrent states that communicate. Define $\tau_i := \mathbb{E}_i T_i$ and $T_i^* := \inf\{n \geq T_j : X_n = i\}$, the first time that the chain visits state i after its first visit to state j, and $T_j^* := \inf\{n \geq T_i : X_n = j\}$.

✠ (2.1)

(i) [10 points] Explain why $\mathbb{P}_i\{T_j = \infty\} = 0$. Hint: What do you know about the probability $\theta_j := \mathbb{P}_i\{T_j < T_i\}$?

(ii) [15 points] Explain why $\mathbb{E}_i T_i^* = \sum_{n \in \mathbb{N}} (n + \mathbb{E}_j T_i) \mathbb{P}_i\{T_j = n\}$. Hint: Condition.

(iii) [10 points] Deduce that $\mathbb{E}_i T_i^* = \mathbb{E}_i T_j + \mathbb{E}_j T_i$.

(iv) [5 points] Deduce that $\mathbb{E}_i T_i^* = \mathbb{E}_j T_j^*$.

✠ (2.2) Let S_k denote the length of the kth excursion from state i. That is, $S_1 = T_i$, $S_1 + S_2$, $S_1 + S_2 + S_3$, ... are the successive times at which the chain returns to state i. Let V denote the first excursion during which the chain passes through state j.

(i) [5 points] Explain why V has a geometric distribution with $\mathbb{E}_i V < \infty$.

(ii) [5 points] Explain why $\mathbb{E}_i S_k = \tau_i$ for each k.

(iii) [5 points] Explain why $T_i^* \leq \sum_{k \in \mathbb{N}} S_k 1\{V \geq k\}$.

(iv) [10 points] Explain why S_k is independent of the event $\{V \geq k\}$

(v) [15 points] Deduce that $\mathbb{E}_i T_i^* \leq \tau_i \mathbb{E}_i V$. Hint: What does $\sum_{k \in \mathbb{N}} 1\{V \geq k\}$ count?

✠ (2.3) Suppose state i is positive recurrent, that is, $\tau_i < \infty$.

(i) [10 points] Use the results from the first two problems to show that $\mathbb{E}_i T_j < \infty$ and $\mathbb{E}_j T_i < \infty$.

(ii) [10 points] Deduce that state j is also positive recurrent.

✠ (2.4) Let $\{X_n : n = 0, 1, 2, \ldots\}$ be an irreducible Markov Chain on a state space $\mathcal{S}$, with period 2. Define $Y_n = X_{2n}$ and $Z_n = X_{2n+1}$ for $n = 0, 1, 2, \ldots$. Choose and hold fixed some state i_0. For each state j define $N_j := \{n : \mathbb{P}_{i_0}\{X_n = j\} > 0\}$.

(i) [10 points] Show that N_j cannot contain both odd and even integers.

Let $\mathcal{S}_0$ consist of those states j for which N_j contains only even integers, and $\mathcal{S}_1$ consist of those states j for which N_j contains only odd integers.

(ii) [10 points] Show that both $\{Y_n\}$ and $\{Z_n\}$ are aperiodic chains. Are they irreducible? Would they be irreducible if restricted to states in $\mathcal{S}_0$ or $\mathcal{S}_1$? Explain.

(iii) [10 points] If $\{X_n\}$ is positive recurrent, show that both $\{Y_n\}$ and $\{Z_n\}$ are also positive recurrent.

Now suppose $\{X_n\}$ has a stationary distribution π.

(iv) [10 points] Show that $\sum_{j \in \mathcal{S}_0} \pi_j = 1/2$. Hint: If $j \in \mathcal{S}_0$, for which i can $P(i, j)$ be strictly positive?

Write π as $\frac{1}{2}\left(\pi^{(0)} + \pi^{(1)}\right)$, where $\pi^{(0)}$ concentrates on $\mathcal{S}_0$ and $\pi^{(1)}$ concentrates on $\mathcal{S}_1$.

(v) [10 points] Show that both $\{Y_n\}$ and $\{Z_n\}$ have $\pi^{(0)}$ and $\pi^{(1)}$ as stationary distributions.

(vi) [10 points] From what you know about the stationary distributions of irreducible, aperiodic chains prove that $\pi^{(0)}$ and $\pi^{(1)}$ are unique. Deduce that $\{X_n\}$ can have only one stationary distribution.

(vii) [15 points] For an arbitrary initial distribution μ, describe the behaviour of $\mathbb{P}_\mu\{X_n = j\}$ for large n. Hint: Consider first the case where μ concentrates at a single i. Consider separately n odd and n even.

(2.5) (Knight's tour) Chang Exercise 1.28. Hint: Reduce to a random walk on a graph.

(2.6) (Stationary probability assigned to a null recurrent state) Chang Exercise 1.30.