Statistics 251/551 **Due: Wednesday 24 March**
2004: Sheet 4

✠ (4.1) [30 points] In Chang Example 4.15 (the "Say Red" game), Joe asserted that $M_n = R_n/(52-n)$ is a martingale. Prove it.

✠ (4.2) [20 points + 20 points] Chang Problem 4.6 (Wright–Fisher).

✠ (4.3) [30 points + 20 points] Chang Problem 4.8 (Polya urn). Note that there are two parts to the question.

✠ (4.4) [30 points + 30 points] Chang Problem 4.12 (trading strategy).

✠ (4.5) Let $X_0, X_1, \ldots$ be a random walk on the integers, starting at $X_0 = 0$, with

$$\mathbb{P}\{X_{n+1} = X_n + 1 | X_n = x\} = p \qquad \text{and} \qquad \mathbb{P}\{X_{n+1} = X_n - 1 | X_n = x\} = q,$$

where $0 < p < 1$ and $p = q = 1$.

(i) [15 points] For $s > 0$ define $H(s) = ps + q/s$. Show that $Z_n = s^{X_n}/H(s)^n$ is a martingale.

(ii) [15 points] For fixed constants $A < 0 < B$, define $\tau = \inf\{n \in \mathbb{N} : X_n = A \text{ or } X_n = B\}$. Show that $\tau \wedge N$ is a stopping time for each positive integer N. Note: $x \wedge y$ means $\min(x, y)$.

(iii) [10 points] Show that $\tau < \infty$ with probability 1. Hint: Think of chickens.

(iv) [10 points] For each $t > 0$, show that there are two distinct roots to the equation $e^t = H(s)$. Call them $s_0(t)$ and $s_1(t)$. Hint: Draw a picture.

(v) [10 points] How do the roots $s_0(t)$ and $s_1(t)$ behave as t tends to zero?

(vi) [15 points] Show that

$$1 = \mathbb{E}\left(\frac{s^B}{H(s)^\tau} 1\{\tau \le N, X_\tau = B\}\right) + \mathbb{E}\left(\frac{s^A}{H(s)^\tau} 1\{\tau \le N, X_\tau = A\}\right) + \text{ remainder},$$

where the remainder tends to zero as N tends to infinity.

(vii) [15 points] For $t > 0$ define $G_A(t) = \mathbb{E}\left(e^{-t\tau} 1\{X_\tau = A\}\right)$ and $G_B(t) = \mathbb{E}\left(e^{-t\tau} 1\{X_\tau = B\}\right)$. Deduce from part (vi) that the equation

$$1 = s^B G_B(t) + s^A G_A(t)$$

has solutions $s = s_0(t)$ and $s = s_1(t)$.

(viii) [10 points] Using (vii), solve for $G_A(t)$ and $G_B(t)$ as functions of $s_0(t)$ and $s_1(t)$.

(ix) [10 points] Check your solution to (viii) by letting t tend to zero then comparing with the absorption probabilities for the Gambler's ruin problem.

(x) [5 points] Express $\mathbb{E}e^{-t\tau}$ as a function of $s_0(t)$ and $s_1(t)$.

(4.6) Establish the inequality

$$(*) \qquad \sum_i p_i \log(p_i/q_i) \ge \frac{1}{2}\left(\sum_i |p_i - q_i|\right)^2,$$

for probabilities **p** and **q**, by the following steps.

(i) Let f be a convex increasing function on $\mathbb{R}^+$ with $0 = f(0) = f'(0)$ and f'' convex. Use the integral form of Taylor's theorem and Jensen's inequality to show that

$$f(x) = \tfrac{1}{2}x^2 \int_0^1 2(1-t) f''(xt)\, dt \ge \tfrac{1}{2}x^2 f''(x/3)$$

(ii) Specialize to the case $f_0(x) = (1+x)\log(1+x) - x$ to show that $f_0(x) \ge \frac{1}{2}x^2/(1+x/3)$.

(iii) Explain why inequality $(*)$ is trivial if $p_i > 0 = q_i$ for some i.

(iv) Write p_i as $q_i(1+x_i)$. Show that $\sum_i q_i x_i = 0$ and $\sum_i q_i |x_i| = \sum_i |p_i - q_i|$.

(v) Invoke the Cauchy-Schwarz inequality to show that

$$\left(\sum_i |p_i - q_i|\right)^2 \le \left(\sum_i q_i \frac{x_i^2}{1 + x_i/3}\right)\left(\sum_i q_i(1 + x_i/3)\right)$$

(vi) Complete the proof.