Statistics 251/551 **Due: Wednesday 7 April**
2004: Sheet 5

Throughout the course I have avoided rigorous arguments when passing to limits. In Joe's notes, he refers to a mysterious result called Bounded Convergence occasionally. Here is the full story.

Suppose $\{X_n : n \in \mathbb{N}\}$ is a sequence of random variables, all living on the same Ω.

(a) [Monotone Convergence] If $0 \le Z_1(\omega) \le Z_2(\omega) \le \dots \uparrow Z(\omega)$ at each ω in a set Ω_0 with $\mathbb{P}\Omega_0 = 1$, then $\mathbb{E}(Z_n) \uparrow \mathbb{E}Z$.

(b) [Dominated Converence] If $Z_n(\omega) \to Z(\omega)$ at each ω in a set Ω_0 with $\mathbb{P}\Omega_0 = 1$, and if $\mathbb{E}\sup_{n\in\mathbb{N}} |Z_n| < \infty$, then $\mathbb{E}(Z_n) \to \mathbb{E}Z$. The second condition is usually checked by showing that there is a nonnegative random variable Y with $|Z_n(\omega)| \le Y(\omega)$ for all n and all ω for which $\mathbb{E}Y < \infty$.

While I am a rigorous mood, I'll give you another result that would have been handy a few times:

(c) [Jensen's inequality] If $\Psi(\cdot)$ is a convex function and X is a random variable (with finite expectation) then $\mathbb{E}\Psi(X) \ge \Psi(\mathbb{E}X)$.

For the first three problems, X is a standard Brownian motion and $a < 0 < b$ are constants.

✠ (5.1) [30 points] For the stopping time $\tau = \inf\{t > 0 : X_t = a \text{ or } b\}$, show that

$$\mathbb{P}\{\tau > n\} \le \mathbb{P} \cap_{i=1}^{n} \{|X_i - X_{i-1}| \le |a| + |b|\} \to 0 \qquad \text{as } n \to \infty.$$

Deduce that $\mathbb{P}\{\tau < \infty\} = 1$.

✠ (5.2) In class I used the martingale $M_t = X_t^2 - t$ to find the expected value of the stopping time τ from the previous problem. I started from the Stopping Time Lemma to show that $0 = \mathbb{E}M_{\tau\wedge n}$ for each n in $\mathbb{N}$. That is,

$$(*) \qquad \mathbb{E}\left(\tau \wedge n\right) = \mathbb{E}X_\tau^2 1_{\{\tau\le n\}} + \mathbb{E}X_n^2 1_{\{\tau > n\}}$$

(i) [10 points] Explain why $0 \le \tau(\omega) \wedge n \uparrow \tau(\omega)$ for every ω.

(ii) [10 points] Explain why $X^2_{\tau(\omega)}(\omega) 1_{\{\tau(\omega)\le n\}} \uparrow X^2_{\tau(\omega)}(\omega) 1_{\{\tau(\omega)<\infty\}}$ for every ω.

(iii) [10 points] Explain why $X_n^2(\omega)1_{\{\tau(\omega)>n\}} \to 0$ at every ω for which $\tau(\omega) < \infty$.

(iv) [10 points] Using the Monotone Convergence and Dominated Convergence theorems, explain rigorously why equality $(*)$ becomes $\mathbb{E}\tau = \mathbb{E}X_\tau^2 1_{\{\tau<\infty\}}$ in the limit as $n \to \infty$.

✠ (5.3) [25 points] Find the constant C for which $X_t^3 - CtX_t$ is a martingale.

✠ (5.4) Let $W_1, W_2, \dots, W_N$ be random variables with $W_i \sim N(0, \sigma_i^2)$. Show that

$$\mathbb{E}\max_{i\le N} |W_i| \le \left(\max_{i\le N} \sigma_i\right) \sqrt{2\log(2N)}$$

by following these steps.

(i) [10 points] For fixed $t > 0$, show that

$$\exp\left(t\mathbb{E}\max_{i\le N} |W_i|\right) \le \mathbb{E}\max_{i\le N} e^{t|W_i|} \le \sum\nolimits_{i\le N} \mathbb{E}e^{t|W_i|}$$

Hint: Invoke Jensen's inequality.

(ii) [10 points] Show that

$$\mathbb{E}e^{t|W_i|} \le \mathbb{E}e^{tW_i} + \mathbb{E}e^{-tW_i} \le 2\exp\left(\tfrac{1}{2}t^2\sigma^2\right) \qquad \text{where } \sigma = \max_{i\le N}\sigma_i$$

(iii) [10 points] Deduce from (i) and (ii) that

$$\mathbb{E}\max_{i\le N} |W_i| \le \frac{\log(2N)}{t} + \tfrac{1}{2}\sigma^2 t \qquad \text{for each } t > 0.$$

(iv) [10 points] Choose a t to give the asserted bound.

✠ (5.5) [20+20 points] Chang Problem 5.17.