Statistics 251/551
2004: Solutions to sheet 1

(1.2) I'll write hh for the initial configuration of the stack, th for the configuration with the top coin facing tail up and the bottom coin facing head up, and so on. There are only four possible outcomes for the first two shuffles:

outcome	stack changes	probability	H_1, H_2	$\mathbb{P}\{H_3 = 2 \mid \omega_i\}$
ω_1	$hh \to th \to hh$	θ_1^2	1,2	0
ω_2	$hh \to th \to th$	$\theta_1\theta_2$	1,1	θ_1
ω_3	$hh \to tt \to ht$	$\theta_2\theta_1$	0,1	0
ω_4	$hh \to tt \to hh$	θ_2^2	0,2	0

For part (i):

$$\mathbb{P}\{H_3 = 2 \mid H_1 = 1, H_2 = 1\} = \mathbb{P}\{H_3 = 2 \mid \omega_2\} = \theta_1.$$

For part (ii):

$$\begin{aligned}\mathbb{P}\{H_3 = 2 \mid H_2 = 1\} &= \mathbb{P}\{H_3 = 2 \mid \omega_2 \text{ or } \omega_3\}\\ &= \mathbb{P}\{H_3 = 2 \mid \omega_2\}\mathbb{P}\{\omega_2 \mid \omega_2 \text{ or } \omega_3\} + \mathbb{P}\{H_3 = 2 \mid \omega_3\}\mathbb{P}\{\omega_3 \mid \omega_2 \text{ or } \omega_3\}\\ &= \theta_1 \frac{\theta_1\theta_2}{\theta_1\theta_2 + \theta_1\theta_2} + 0 = \tfrac{1}{2}\theta_1\end{aligned}$$

(1.3) You should draw the state space picture, which shows that returns to state 1, when starting in state 1, can occur only in $n = 5k + 2\ell$ steps, for $k = 1, 2, 3, \ldots$ and $\ell = 0, 1, 2, \ldots$, corresponding to k completed loops around $1 \to 2 \to 3 \to 4 \to 5 \to 1$ with ℓ loops aound $4 \to 5 \to 4$. The choice $k = 1$ gives all the odd values $5, 7, 9, 11, 13, \ldots$. The choice $k = 2$ gives all the even values $10, 12, 14, \ldots$. larger values of k generate no new values for n. Thus

$$\{n \in \mathbb{N} : \mathbb{P}_1\{X_n = 1\} > 0\} = \{5, 7\} \cup \{n \in \mathbb{N} : n \geq 9\}$$

(1.4) Consider a slightly more general form of the problem, with state space $\mathbb{Z}$, the set of all integers, and transition probabilities:

i = state	...	-3	-2	-1	0	1	2	3	...
$P(i, i+1)$	...	θ	θ	θ	1/2	$1-\theta$	$1-\theta$	$1-\theta$	...
$P(i, i-1)$	...	$1-\theta$	$1-\theta$	$1-\theta$	1/2	θ	θ	θ	...

for some $1/2 < \theta < 1$. The homework problem had $\theta = 0.6$.

The equations for the stationary distribution reduce to $\pi_j = P(j+1, j)\pi_{j+1} + P(j-1, j)\pi_{j-1}$ for all j, that is,

$$\pi_0 = \theta\left(\pi_{-1} + \pi_1\right)$$
$$\pi_1 = \tfrac{1}{2}\pi_0 + \theta\pi_2 \quad \text{and} \quad \pi_{-1} = \tfrac{1}{2}\pi_0 + \theta\pi_{-2}$$
$$\pi_2 = (1-\theta)\pi_1 + \theta\pi_3 \quad \text{and} \quad \pi_{-2} = (1-\theta)\pi_{-1} + \theta\pi_{-3}$$
$$\pi_3 = (1-\theta)\pi_2 + \theta\pi_4 \quad \text{and} \quad \pi_{-3} = (1-\theta)\pi_{-2} + \theta\pi_{-4}$$
$$\ldots$$

The question asks for *the* stationary distribution. In fact, for an irreducible chain there can be at most one stationary distribution. The symmetry of the situation suggests we should look for a solution with $\pi_i = \pi_{-i}$ for all $i \geq 1$. If we find such a solution, we are done. If we did't find a solution, then we could either start searching for a solution without symmetry or give an argument why the solution, if it exists, has to be symmetric.

A symmetric solution would satisfy

$$\begin{aligned}
&\pi_0 = 2\theta\pi_1 \\
&\pi_1 = \tfrac{1}{2}\pi_0 + \theta\pi_2 \qquad \text{that is. } \pi_2 = \alpha\pi_1 \text{ where } \alpha = (1-\theta)/\theta = 1/\theta - 1 \\
&\pi_2 = (1-\theta)\pi_1 + \theta\pi_3 \qquad \text{that is, } \pi_3 = -\alpha\pi_1 + (1/\theta)\pi_2 = \alpha^2\pi_1 \\
&\pi_3 = (1-\theta)\pi_2 + \theta\pi_4 \qquad \text{that is, } \pi_4 = -\alpha\pi_2 + (1/\theta)\pi_3 = \alpha^3\pi_1 \\
&\quad \ldots
\end{aligned}$$

The pattern seems clear. A formal inductive proof would ensure that we are not being fooled by a few cases.

[[Alternatively, we could rearrange the equation for π_2 into $(1-\theta)(\pi_2 - \pi_1) = \theta(\pi_3 - \pi_2)$, that is,

$$\pi_3 - \pi_2 = \alpha(\pi_2 - \pi_1),$$

then work our way down the following equations:

$$\begin{aligned}
&\pi_4 - \pi_3 = \alpha(\pi_3 - \pi_2) = \alpha^2(\pi_2 - \pi_1) \\
&\pi_5 - \pi_4 = \alpha(\pi_4 - \pi_3) = \alpha^2(\pi_3 - \pi_2) = \alpha^3(\pi_2 - \pi_1) \\
&\ldots
\end{aligned}$$

The sum telescopes when we add, leaving

$$\pi_{k+1} - \pi_1 = \sum\nolimits_{i=1}^{k}(\pi_{i+1} - \pi_i) = (1 + \alpha + \alpha^2 + \ldots + \alpha^{k-1})(\pi_2 - \pi_1)$$

We must have $\pi_{k+1} \to 0$ as $k \to \infty$, for otherwise $\sum_{i\in\mathbb{Z}} \pi_i$ would not converge. The limiting form of the previous equation is

$$-\pi_1 = (1-\alpha)^{-1}(\pi_2 - \pi_1),$$

which again gives $\pi_2 = \alpha\pi_1$ and

$$\pi_{k+1} = \pi_1 + (1 + \alpha + \alpha^2 + \ldots + \alpha^{k-1})(\alpha - 1)\pi_1 = \alpha^k\pi_1$$

Notice that this argument does not depend on the assumption of symmetry. It could equally well be applied on the other side of the origin, giving $\pi_{-(k+1)} = \alpha^k\pi_{-1}$. The equations for $\pi_{\pm 1}$ woud then give

$$\pi_1(1 - \theta\alpha) = \tfrac{1}{2}\pi_0 = \pi_{-1}(1 - \theta\alpha),$$

which leads us back to the symmetry property.]]

Finally, the requirement that the π_i's sum to 1 lets us solve for π_1:

$$1 = \sum\nolimits_{i\in\mathbb{Z}} \pi_1 = \pi_0 + 2\sum\nolimits_{i\geq 1} \pi_i = \left(2\theta + 2(1-\alpha)^{-1}\right)\pi_1,$$

that is, $\pi_1 = (2\theta - 1)/(2\theta)^2$. For $\theta = 0.6$ we have

$$\pi_0 = \tfrac{1}{6} \qquad \text{and} \qquad \pi_{\pm k} = \tfrac{5}{36}\left(\tfrac{2}{3}\right)^{k-1} \qquad \text{for } k = 1, 2, \ldots$$

For the general solution, we needed $1/2 < \theta$ to ensure that $\alpha < 1$ and $\sum_k \alpha^k < \infty$. For $\theta = 1/2$ the method breaks down, as we know it must: there is no stationary probability distribution for the symmetric random walk on the integers.

(1.5) We have

$$\mathbb{P}_\mu\{X_{n+1} = j\} = \sum\nolimits_{i\in\mathcal{S}} \mathbb{P}_\mu\{X_n = i\}P(i,j) \geq \sum\nolimits_{i\in\mathcal{S}_0} \mathbb{P}_\mu\{X_n = i\}P(i,j)$$

for every finite subset $\mathcal{S}_0$ of $\mathcal{S}$. It is legitimate to pass to the limit inside a finite sum. Thus

$$\pi_j \geq \sum\nolimits_{i\in\mathcal{S}_0} \pi_i P(i,j) \qquad \text{for finite } \mathcal{S}_0 \subseteq \mathcal{S}.$$

Take the limit as $\mathcal{S}_0 \uparrow \mathcal{S}$ to deduce that $\pi_j \geq \sum_{i\in\mathcal{S}} \pi_i P(i,j)$, for each j. That is, for some $\delta_j \geq 0$,

$$\pi_j = \delta_j + \sum\nolimits_{i\in\mathcal{S}_0} \pi_i P(i,j)$$

Sum over j, using the fact that the order of summation can be changed when summing nonnegative quantities, to deduce that

$$1 = \sum\nolimits_{j\in\mathcal{S}} \pi_j = \sum\nolimits_{j\in\mathcal{S}} \delta_j + \sum\nolimits_{i\in\mathcal{S}} \pi_i \sum\nolimits_{j\in\mathcal{S}} P(i,j).$$

Then use the fact that $\sum_{j\in\mathcal{S}} P(i,j) = 1$ to conclude that $\delta_j = 0$ for every j.