# Solution to Problem 2.4

We start from an irreducible Markov chain $\{X_n : n = 0, 1, 2, \ldots\}$ on a state space $\mathcal{S}$ with transition probabilities $P(i, j)$. The chain has period 2 and a stationary distribution $\pi$.

Many of you had trouble relating the $X_n$ and $Y_n$ chains, confusing the state space with the chains themselves and failing to distinguish between transition probabilities for the $X_n$ and $Y_n$ chains. Remember that irreducibility and periodicity are properties of $P$ and $\mathcal{S}$. To emphasize these facts, I will write out a solution in matrix notation, which you could reinterpret via summations if $\mathcal{S}$ is infinite.

A state $i_0$ is chosen arbitrarily from $\mathcal{S}$. Then for each $j$ in $\mathcal{S}$ we define

$$N_j = \{n \in \mathbb{N} : P^n(i_0, j) > 0\}$$

When $j$ equals $i_0$, the set $N_{i_0}$ corresponds to the set used to define the period of the state $i_0$: that is, $2 = \gcd N_{i_0}$. For $j \neq i_0$, the set $N_j$ is not the one used to define the period of state $j$.

To show that all elements of an $N_j$ have the same parity (that is, all are odd or all are even), argue from existence (by irreducibility) of an $m$ such that $P^m(j, i_0) > 0$. If $n$ and $n'$ are both in $N_j$, then $P^{n+m}(i_0, i_0) > 0$ and $P^{n'+m}(i_0, i_0) > 0$, which implies that both $n + m$ and $n' + m$ are even numbers. By subtraction, $n - n'$ is also even, forcing $n$ and $n'$ to have the same parity.

Remark. Some of you incorrectly asserted that $P^{|n-n'|}(i_0, i_0) > 0$. If there are paths of length 3 and length 5 leading from $i_0$ to $j$, how could they be combined to get a path of length 2 from $i_0$ to $i_0$?

We define $\mathcal{S}_0$ as the set of states $j$ for which all elements of $N_j$ are even, and $\mathcal{S}_1$ as the set of states $j$ for which all elements of $N_j$ are odd. If two states are both in $\mathcal{S}_0$ or are both in $\mathcal{S}_1$, say that they have the same parity; otherwise, say they have opposite parity.

It is easy to show that a pair of states $(i, j)$ for which $P(i, j) > 0$ must have opposite parity: if $n \in N_i$ then $P^n(i_0, i) > 0$, whence $P^{n+1}(i_0, j) > 0$ and $n + 1 \in N_j$.

If we label the states so that all those in $\mathcal{S}_0$ precede those in $\mathcal{S}_1$, the transition matrix $P$ takes the form

$$\begin{array}{cc} & \begin{array}{cc} \mathcal{S}_0 & \mathcal{S}_1 \end{array} \\ \begin{array}{c} \mathcal{S}_0 \\ \mathcal{S}_1 \end{array} & \begin{pmatrix} 0 & P_1 \\ P_0 & 0 \end{pmatrix} \end{array}$$

The Markov chain $Y_n := X_{2n}$ has transition matrix

$$Q = P^2 = \begin{pmatrix} Q_0 & 0 \\ 0 & Q_1 \end{pmatrix} \qquad \text{where } Q_0 = P_1 P_0 \text{ and } Q_1 = P_0 P_1$$

with $n$-step transition probabilities

$$Q^n = \begin{pmatrix} Q_0^n & 0 \\ 0 & Q_1^n \end{pmatrix}$$

Clearly $\mathbb{P}\{Y_n = j \mid Y_0 = i\} = 0$ for all $n$ if $i$ and $j$ have opposite parity; $Y_n$ is not irreducible as a chain on the state space $\mathcal{S}$. If $Y_n$ starts in $\mathcal{S}_0$, it stays there forever; if $Y_n$ starts in $\mathcal{S}_1$, it stays there forever.

If we start $Y_n$ somewhere in $\mathcal{S}_0$, we may think of it as a Markov chain with state space $\mathcal{S}_0$ and transition matrix $Q_0$. Suppose $i$ and $j$ are two states in $\mathcal{S}_0$. From the definition of $N_i$ and $N_j$ (and the fact that $i_0$ has period 2 for the transition matrix $P$), there exist even integers $2m, 2n, 2m', 2n'$ such that

$$\begin{array}{ccc} P^{2n}(i_0, i) > 0 & \text{and} & P^{2m}(i, i_0) > 0 \\ P^{2n'}(i_0, j) > 0 & \text{and} & P^{2m'}(j, i_0) > 0 \end{array}$$

It follows that

$$Q_0^{m+n'}(i, j) = P^{2m+2n'}(i, j) > 0$$

Thus, $Q_0$ defines an irreducible chain on $\mathcal{S}_0$. The fact that $i_0$ has period 2 for the $X_n$ chain implies that

$$Q^n(i_0, i_0) = P^{2n}(i_0, i_0) > 0 \qquad \text{for all } n \text{ large enough.}$$

That is, the chain on $\mathcal{S}_0$ with transition matrix $Q_0$ has period 1.

Similar reasoning shows that the chain on $\mathcal{S}_1$ with transition matrix $Q_1$ is irreducible with period 1.

**Stationary distributions**

By definition of $\pi$ as a stationary distribution for the $X_n$ chain,

$$\pi_j = \sum\nolimits_{i\in\mathcal{S}} \pi_i P(i,j) \qquad \text{for all } i,j \in \mathcal{S}.$$

If $j \in \mathcal{S}_0$ we have $P(i,j) = 0$ for all $i \in \mathcal{S}_0$. And if $i \in \mathcal{S}_1$ we have $\sum_{j\in\mathcal{S}_0} P(i,j) = 1$. Thus

$$\sum\nolimits_{j\in\mathcal{S}_0} \pi_j = \sum\nolimits_{j\in\mathcal{S}_0}\sum\nolimits_{i\in\mathcal{S}_1} \pi_i P(i,j) = \sum\nolimits_{i\in\mathcal{S}_1} \pi_i \sum\nolimits_{j\in\mathcal{S}_0} P(i,j) = \sum\nolimits_{i\in\mathcal{S}_1} \pi_i$$

That is, $\pi$ gives the same probability to $\mathcal{S}_0$ and to $\mathcal{S}_1$. The fact that $1 = \pi(\mathcal{S}) = \pi(\mathcal{S}_0) + \pi(\mathcal{S}_1)$ then gives $\pi(\mathcal{S}_0) = \pi(\mathcal{S}_1) = 1/2$.

We define probability distributions $\pi^{(0)}$ as 2 times the restriction of $\pi$ to $\mathcal{S}_0$ and $\pi^{(1)}$ as 2 times the restriction of $\pi$ to $\mathcal{S}_1$. The stationarity property $\pi = \pi P$ becomes

$$\left(\tfrac{1}{2}\pi^{(0)}, \quad \tfrac{1}{2}\pi^{(1)}\right)\begin{pmatrix} 0 & P_1 \\ P_0 & 0 \end{pmatrix} = \left(\tfrac{1}{2}\pi^{(0)}, \quad \tfrac{1}{2}\pi^{(1)}\right).$$

that is, $\pi^{(1)}P_0 = \pi^{(0)}$ and $\pi^{(0)}P_1 = \pi^{(1)}$, from which it follows that

$$\pi^{(0)}Q_0 = \pi^{(0)}P_1P_0 = \pi^{(1)}P_0 = \pi^{(0)}$$
$$\pi^{(1)}Q_1 = \pi^{(1)}\pi^{(0)}P_0P_1 = \pi^{(0)}P_1 = \pi^{(1)}$$

The $Y_n$ chains on $\mathcal{S}_0$ and $\mathcal{S}_1$ have stationary distributions $\pi^{(0)}$ and $\pi^{(1)}$.

By the BLT, these stationary distributions for $Q_0$ and $Q_1$ are unique, which implies that $\pi$ must be the unique stationary distribution for $P$. We can say even more. As $n \to \infty$,

$$Q_0^n(i,j) \to \pi_j^{(0)} \qquad \text{for all } i,j \in \mathcal{S}_0$$
$$Q_1^n(i,j) \to \pi_j^{(1)} \qquad \text{for all } i,j \in \mathcal{S}_1$$

That is,

$$P^{2n}(i,j) \to \begin{cases} \pi_j^{(0)} & \text{if } i,j \in \mathcal{S}_0 \\ 0 & \text{if } i \in \mathcal{S}_0 \text{ and } j \in \mathcal{S}_1 \\ \pi_j^{(1)} & \text{if } i,j \in \mathcal{S}_1 \\ 0 & \text{if } i \in \mathcal{S}_1 \text{ and } j \in \mathcal{S}_0 \end{cases}.$$

Similarly, the relationship

$$P^{2n+1} = \begin{pmatrix} Q_0^n & 0 \\ 0 & Q_1^n \end{pmatrix}\begin{pmatrix} 0 & P_1 \\ P_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & Q_0^n P_1 \\ Q_1^n P_0 & 0 \end{pmatrix}$$

implies

$$P^{2n+1}(i,j) \to \begin{cases} 0 & \text{if } i,j \in \mathcal{S}_0 \\ \pi_j^{(1)} & \text{if } i \in \mathcal{S}_0 \text{ and } j \in \mathcal{S}_1 \\ 0 & \text{if } i,j \in \mathcal{S}_1 \\ \pi_j^{(0)} & \text{if } i \in \mathcal{S}_1 \text{ and } j \in \mathcal{S}_0 \end{cases}.$$

For a general initial distribution $\mu$ for the $X_n$ chain, define $\theta_0 = \mu(\mathcal{S}_0)$ and $\theta_1 = \mu(\mathcal{S}_1)$. By summing over contributions from $\mathcal{S}_0$ and $\mathcal{S}_1$ we then get a description of the limiting behavior of the chain:

$$\mathbb{P}_\mu\{X_{2n} = j\} \to \begin{cases} \theta_0\pi_j^{(0)} & \text{if } j \in \mathcal{S}_0 \\ \theta_1\pi_j^{(1)} & \text{if } j \in \mathcal{S}_1 \end{cases}.$$

and

$$\mathbb{P}_\mu\{X_{2n+1} = j\} \to \begin{cases} \theta_1\pi_j^{(0)} & \text{if } j \in \mathcal{S}_0 \\ \theta_0\pi_j^{(1)} & \text{if } j \in \mathcal{S}_1 \end{cases}.$$

Notice that a choice with $\theta_0 = 1/2$ leads us to a limiting value $\pi_j$ for $\mathbb{P}_\mu\{X_n = j\}$, for all $j$.