Statistics 251/551
2004: Solutions to sheet 4

(4.1) Write $\mathcal{F}_n$ for the information given by $R_1, R_2, \ldots, R_n$. If all initial orderings of the deck were equally likely, then all possible orderings of the unseen cards are equally likely. In particular, the conditional probability of the top card being red is $R_n/(52-n)$, the proportion of red cards remaining in the deck. Thus

$$\mathbb{P}\big(R_{n+1} = R_n - 1 \mid \mathcal{F}_n\big) = R_n/(52-n)$$
$$\mathbb{P}\big(R_{n+1} = R_n \mid \mathcal{F}_n\big) = 1 - R_n/(52-n)$$

and

$$\mathbb{E}\big(R_{n+1} \mid \mathcal{F}_n\big) = \frac{(R_n-1)R_n + R_n(52-n-R_n)}{52-n} = \frac{R_n(51-n)}{52-n}.$$

Divide through by $51-n$ to get the martingale property, $\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) = M_n$.

Most of you noticed that R_n is Markov chain, so that conditioning on $\mathcal{F}_n$ can be reduced to conditioning on R_n. Some of you forgot to distinguish between conditional and unconditional calculations.

(4.2) Write $\mathcal{F}_n$ for the information given by $X_0, X_1, \ldots, X_n$. The sequence $\{X_n : n = 0, 1, \ldots\}$ is a Markov chain, with the conditional distribution of X_{n+1} given X_n being Binomial$(d, X_n/d)$. Consequently,

$$\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = d(X_n/d) = X_n.$$

That is, X_n is a martingale.

There are several ways to use the martingale property to find the probability θ_W that the population is eventually all white.

A crude argument: $\mathbb{E}X_n$ stays constant at its initial value x_0. For very large n, this value should be close to $0(1-\theta_W) + d\theta_W$. [Waving of hands follows.]

I was hoping for a more formal argument involving stopping times and the STL. For example, you could define $\tau = \inf\{n : X_n = 0 \text{ or } d\}$. You could use the chicken argument to show that $\tau < \infty$ with probability one: $\mathbb{P}\{X_{n+1} = d \mid X_n = i\} \geq d^{-d}$ for every i with $0 < i < d$. By the STL, for each positive integer N,

$$x_0 = \mathbb{E}X_0 = \mathbb{E}X_{\tau\wedge N} = 0\mathbb{P}\{X_\tau = 0, \tau \leq N\} + d\mathbb{P}\{X_\tau = d, \tau \leq N\} + \mathbb{E}X_N\mathbb{I}\{\tau > N\}$$

The last term is bounded by $d\mathbb{P}\{\tau > N\}$ which tends to zero as N tends to ∞. The other two terms also converge. In the limit we have

$$x_0 = d\mathbb{P}\{X_\tau = d, \tau < \infty\} = d\theta_W,$$

as suggested by the crude argument.

(4.3) Consider the general problem where, after each draw, we replace the selected ball by $c+1$ balls of the same color and d balls of the other color. Let W_n be the number of white balls in the urn after completion of the nth step and B_n be the number of black balls. Notice that $N_n := W_n + B_n = w + b + n(c+d)$ if we start with $W_0 = w$ white balls and $B_0 = b$ black balls.

The history up to completion of the nth step, $\mathcal{F}_n$, tells us the values of W_i and B_i for $i \leq n$. In fact we only need the value of W_n (because N_n is deterministic) to specify the conditional distribution,

$$W_{n+1} \mid \mathcal{F}_n = \begin{cases} W_n + c & \text{with probability } M_n \\ W_n + d & \text{with probability } 1 - M_n \end{cases} \qquad \text{where } M_n = W_n/N_n.$$

Thus

$$\mathbb{E}(M_{n+1} \mid \mathcal{F}_n) = \frac{(W_n+c)M_n + (W_n+d)(1-M_n)}{N_{n+1}} = \frac{N_nM_n + (c-d)M_n + d}{N_{n+1}} = M_n\frac{N_n+c-d}{N_{n+1}} + \frac{d}{N_{n+1}}.$$

If $d = 0$ then $N_n + c = N_{n+1}$ and the last expression reduces to M_n. The M_n process is then a martingale. More generally, the last expression equals M_n if and only if

$$\frac{d}{N_{n+1}} = M_n\frac{N_{n+1} - (N_n + c - d)}{N_{n+1}} = M_n\frac{2d}{N_{n+1}}$$

If $d \neq 0$, the right-hand side equals M_n only when $M_n = 1/2$. As $\mathbb{P}\{M_n = 1/2\} < 1$ for $n \geq 1$, the process is not a martingale.

(4.4) For part (a) we need to solve simple linear equations to get the averaging property for a martingale. The solutions are $a = 5.5$ and $b = 10$ and $c = 26$. The trading strategy is

$$H_0 \equiv 2 \qquad \text{and} \qquad H_1 = 4 \times \mathbb{I}\{S_1 = 6\} + \tfrac{1}{2} \times \mathbb{I}\{S_1 = 2\}$$

(4.5) By direct calculation for the conditional distribution of X_{n+1} given $\mathcal{F}_n$,

$$\mathbb{E}(s^{X_{n+1}} \mid \mathcal{F}_n) = s^{X_n}\mathbb{E}\left(s^{X_{n+1}-X_n} \mid \mathcal{F}_n\right) = s^{X_n}\left(ps^1 + qs^{-1}\right) = s^{X_n}H(s).$$

Divide both sides by $H(s)^{n+1}$ to deduce that Z_n is martingale.

For each i in $\mathbb{N}_0$,

$$\{\tau \wedge N = i\} = \begin{cases} \emptyset & \text{if } N < i \\ \{\tau \geq i\} & \text{if } N = i \\ \{\tau = i\} & \text{if } N > i \end{cases}$$

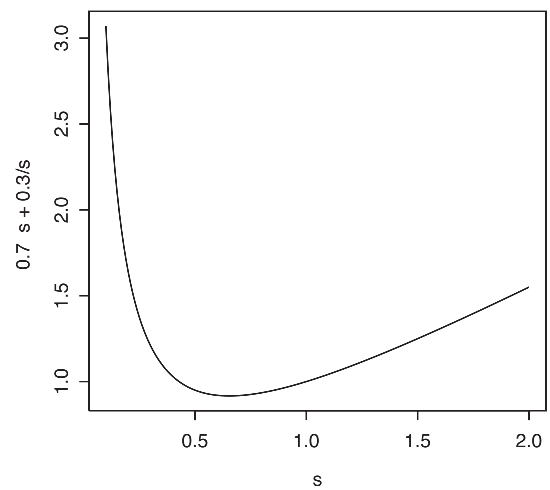
In each case, the event on the right-hand side depends only on $\mathcal{F}_i$ information.

Define $K := |A| + B$. For each n,

$$\mathbb{P}\{\tau \leq n + K \mid \tau > n\} \geq p^{|A|+B},$$

because K successive outcomes with $X_{i+1} = X_i + 1$ would surely cause the X process to hit B if it hadn't already hit A. Deduce that $\mathbb{P}\{\tau > nK\}$ converges to zero geometrically fast.

A plot of $H(s)$ versus s, for the special case $p = 0.7$, looks like:



The picture for general p looks similar, with the minimum value of $\sqrt{4pq}$ occurring when $s = \sqrt{q/p}$. Notice that $4pq \leq 1$, with equality only when $p = 1/2$. For $t > 0$ we have $e^t > 1$, so that the equation $e^t = H(s)$ has two distinct, positive roots,

$$\left(e^t \pm \sqrt{e^{2t} - 4pq}\right)/(2p)$$

Write $s_0(t)$ for the smaller root and $s_1(t)$ for the larger. The roots decrease to the two roots of the equation $H(s) = 1$, namely $s = 1$ and $s = q/p$, as t decreases to 0. Notice that the new roots are distinct if $p \neq 1/2$, but they both equal 1 if $p = 1/2$.

For a fixed natural number N, the stopping time $\tau \wedge N$ is bounded above by the constant N. The STL gives,

$$1 = \mathbb{E}Z_0 = \mathbb{E}Z_{\tau\wedge N} = \mathbb{E}\left(\frac{s^\tau}{H(s)^\tau}\mathbb{I}\{\tau \leq N\}\right) + \mathbb{E}\left(Z_N\mathbb{I}\{\tau > N\}\right)$$

Remember that we are only interested in values of s for which $H(s) = e^t > 1$. On the set $\{\tau > N\}$ the random variable X_N lies between A and B, and $0 \leq Z_N \leq \max(s^B, s^A)/H(s)^N$. The last term in the previous display is smaller than $\mathbb{P}\{\tau > N\}\max(s^B, s^A)/H(s)^N$, which to zero as N tends to ∞. The first term splits into

$$\mathbb{E}\left(\frac{s^B}{H(s)^\tau}\mathbb{I}\{\tau \le N, X_\tau = B\}\right) + \mathbb{E}\left(\frac{s^A}{H(s)^\tau}\mathbb{I}\{\tau \le N, X_\tau = A\}\right)$$

which converges to

$$s^B\mathbb{E}\left(H(s)^{-\tau}\mathbb{I}\{X_\tau = B\}\right) + s^A\mathbb{E}\left(H(s)^{-\tau}\mathbb{I}\{X_\tau = A\}\right).$$

In particular, for s equal to either of the two roots to the equation $H(s) = e^t$, the limiting form of the equality from the STL is

$$1 = s^B G_B(t) + s^A G_A(t) \qquad \text{where } G_A(t) = \mathbb{E}\left(e^{-t\tau}1\{X_\tau = A\}\right) \text{ and } G_B(t) = \mathbb{E}\left(e^{-t\tau}1\{X_\tau = B\}\right).$$

That is,

$$\begin{aligned} 1 &= s_0(t)^B G_B(t) + s_0(t)^A G_A(t) \\ 1 &= s_1(t)^B G_B(t) + s_1(t)^A G_A(t) \end{aligned}$$

The fact that $s_0(t) \ne s_1(t)$ allows us to solve the equations, giving

$$G_A(t) = \frac{s_1(t)^B - s_0(t)^B}{s_0(t)^A s_1(t)^B - s_0(t)^B s_1(t)^A} = \frac{s_0(t)^{-B} - s_1(t)^{-B}}{s_0(t)^{A-B} - s_1(t)^{A-B}}$$

and

$$G_B(t) = \frac{s_1(t)^A - s_0(t)^A}{s_0(t)^B s_1(t)^A - s_0(t)^A s_1(t)^B} = \frac{s_0(t)^{-A} - s_1(t)^{-A}}{s_0(t)^{B-A} - s_1(t)^{B-A}}$$

Note that it was important to have two distinct roots to ensure that

$$s_0^A s_1^B - s_0^B s_1^A = s_0^{A+B}\left((s_1/s_0)^B - (s_1/s_0)^A\right) \ne 0.$$

If we put t equal to 0 with $p = 1/2$ we would not get a unique solution to the equations, both of which would reduce to $1 = G_A(0) + G_B(0)$.

For $p > 1/2$, we know that $s_0(t) \to q/p$ and $s_1(t) \to 1$, implying

$$\mathbb{P}\{\text{ walk hits } A \text{ before } B\ \} = \lim_{t\to 0} G_A(t) = \frac{1 - (q/p)^B}{(q/p)^A - (q/p)^B},$$

the solution obtained in Lecture 14 (Wednesday 25 Feb). The same limit appears when $p < 1/2$. If $p = 1/2$, we have a slight difficulty, in that the ratio for $G_A(t)$ tends to $0/0$. However, writing $g_k(s)$ for s^{-k} we have

$$\lim_{s_0\to 1, s_1\to 1} \frac{g_B(s_0) - g_B(s_1)}{g_{B-A}(s_0) - g_{B-A}(s_1)} = \frac{g_B'(1)}{g_{B-A}'(1)} = \frac{B}{B-A},$$

again agreeing with the solution from the Lecture. Finally,

$$\mathbb{E}e^{-t\tau} = G_A(t) + G_B(t) = \frac{s_1(t)^B - s_0(t)^B - s_1(t)^A + s_0(t)^A}{s_0(t)^A s_1(t)^B - s_0(t)^B s_1(t)^A} \qquad \text{for } t > 0.$$

There is not much that can conveniently be extracted from this mess; but, in theory, it uniquely determines the distribution of τ.