LOOKING BACK — FINDING A PARADOX

When we look back on the Brownian bridge representations as a stochastic integral (9.13), or as a function of standard Brownian motion (9.11), we see the seeds of a paradox. In the integral representation (9.13), we see that X_t depends on the Brownian path up to time t, but in the function representation (9.11) we seem to see that X_t depends on B_1.

This may seem strange, but there is nothing wrong. The processes

$$X_t = (1-t)\int_0^t \frac{1}{1-s}\,dB_s \text{ and } X'_t = B_t - tB_1$$

are both honest Brownian bridges. They can even be viewed as elements of the same probability space; nevertheless, the processes are not equal, and there is no reason to be surprised that the two filtrations

$$\mathcal{F}_t = \sigma\{\,X_s\colon 0 \le s \le t\,\} \text{ and } \mathcal{F}'_t = \sigma\{\,X'_s\colon 0 \le s \le t\,\}$$

are also different. For an even more compelling example, consider the process X''_t defined by taking $X''_t = X'_{1-t}$. The process X''_t is just one more Brownian bridge, yet its natural filtration is completely tangled up with that of X'_t!

9.4. Existence and Uniqueness Theorems

In parallel with the theory of ordinary differential equations, the theory of SDEs has existence theorems that tell us when an SDE must have a solution, and there are uniqueness theorems that tell us when there is at most one solution. Here, we will consider only the most basic results.

THEOREM 9.1 (Existence and Uniqueness). *If the coefficients of the stochastic differential equation*

(9.14) $$dX_t = \mu(t,X_t)\,dt + \sigma(t,X_t)\,dB_t, \quad \textit{with } X_0 = x_0 \textit{ and } 0 \le t \le T,$$

satisfy a space-variable Lipschitz condition

(9.15) $$|\mu(t,x)-\mu(t,y)|^2 + |\sigma(t,x)-\sigma(t,y)|^2 \le K|x-y|^2$$

and the spatial growth condition

(9.16) $$|\mu(t,x)|^2 + |\sigma(t,x)|^2 \le K(1+|x|^2),$$

then there exists a continuous adapted solution X_t of equation (9.14) that is uniformly bounded in $L^2(dP)$:

$$\sup_{0\le t\le T} E(X_t^2) < \infty.$$

Moreover, if X_t and Y_t are both continuous L^2 bounded solutions of equation (9.14), then

(9.17) $$P(\,X_t = Y_t \textit{ for all } t \in [0,T]\,) = 1.$$

NATURE OF THE COEFFICIENT CONDITION

A little experimentation with ODEs is sufficient to show that some condition such as (9.16) is needed in order to guarantee the existence of a solution of the SDE (9.14). For example, if we take

$$\sigma(t,x) \equiv 0 \text{ and } \mu(t,x) = (\beta-1)^{-1}x^\beta \text{ with } \beta > 1,$$

then our target equation (9.14) is really an ODE dressed up in SDE clothing:

$$dX_t = (\beta-1)^{-1}X_t^\beta\,dt \quad \text{with } X_0 = 1.$$

The unique solution to this equation on the interval $[0,1)$ is given by

$$X_t = (1-t)^{-1/(\beta-1)},$$

so there is no continuous solution to the equation on $[0,T]$ when $T > 1$. What runs amuck in this example is that the coefficient $\mu(t,x) = (\beta-1)^{-1}x^\beta$ grows faster than x, a circumstance that is banned by the coefficient condition (9.16).

PROOFS OF EXISTENCE AND UNIQUENESS

The proof of the uniqueness part of Theorem 9.1 is a bit quicker than the existence part, so we will tackle it first. The plan for the proof of the uniqueness theorem for SDEs can be patterned after one of traditional uniqueness proofs from the theory of ODEs. The basic idea is that the difference between two solutions can be shown to satisfy an integral inequality that has zero as its only solution.

From the representation given by the SDE, we know that the difference of the solutions can be written as

$$X_t - Y_t = \int_0^t \mu(s,X_s)-\mu(s,Y_s)\,ds + \int_0^t \sigma(s,X_s)-\sigma(s,Y_s)\,dB_s,$$

so by the elementary bound $(u+v)^2 \le 2u^2+2v^2$ we find

$$E(|X_t-Y_t|^2) \le 2E\left[\left(\int_0^t \mu(s,X_s)-\mu(s,Y_s)\,ds\right)^2\right] + 2E\left[\left(\int_0^t \sigma(s,X_s)-\sigma(s,Y_s)\,dB_s\right)^2\right].$$

By Cauchy's inequality (and the 1-trick), the first summand is bounded by

(9.18) $$2tE\left[\int_0^t |\mu(s,X_s)-\mu(s,Y_s)|^2\,ds\right],$$

whereas by the Itô isometry the second summand simply equals

(9.19) $$2E\left[\int_0^t |\sigma(s,X_s)-\sigma(s,Y_s)|^2\,ds\right].$$

Here we should note that the use of the Itô isometry is indeed legitimate because the Lipschitz condition (9.15) on σ together with the L^2-boundedness of X_t and Y_t will guarantee that

(9.20) $$|\sigma(s,X_s)-\sigma(s,Y_s)| \in \mathcal{H}^2[0,T].$$

When the estimates (9.18) and (9.19) are combined with the coefficient condition (9.15), we find that for $C = 2K\max(T,1)$ we have

(9.21) $$E(|X_t-Y_t|^2) \le C\int_0^t E(|X_s-Y_s|^2)\,ds < \infty.$$

If we set $g(t) = E(|X_t-Y_t|^2)$, the last equation tells us that

(9.22) $$0 \le g(t) \le C\int_0^t g(s)\,ds \text{ for all } 0 \le t \le T,$$