Statistics 251b/551b, spring 2009
Take-home test 1
Due: Wednesday 4 March

- *You should prepare solutions to the following questions without help or hints from anybody else. In particular, if you have been working in a group you must suspend the group arrangement for the test; you must not discuss the questions with your group buddy.*
- *If you have questions of interpretation, or of clarification of the meaning of a question, ask David Pollard.*
- *Make sure you explain your calculations and notation.*
- *Each part of each question is worth 5 points.*
- *As usual, even if you are unable to solve one part of a question you may still use the result for the following parts.*

[1] For each of the following questions, $\mathcal{S}$ denotes the set $\{0, 1, 2, 3, \dots\} = \{0\} \cup \mathbb{N}$.

(i) Give an example of an irreducible Markov chain with state space $\mathcal{S}$ for which there exists a stationary measure but no stationary probability distibution.

(ii) Give an example of a Markov chain with state space $\mathcal{S}$ that has at least two distinct stationary probability distibutions.

(iii) Give an example of a martingale taking values in $\mathcal{S}$ that is not a Markov chain.

(iv) Give an example of an irreducible, aperiodic Markov chain with state space $\mathcal{S}$ for which $\mathbb{P}_i\{X_n = i\} = 0$ for $1 \le n \le 100$ for every state i.

[2] For a fixed positive integer d let

$$\mathcal{S} = \{(i_1, i_2) \in \mathbb{N} \times \mathbb{N} : 1 \le i_1 \le d \text{ and } 1 \le i_2 \le d\}$$

denote the $d \times d$ lattice of points with integer coordinates running from 1 to d. Let $X_n = (X_{n,1}, X_{n,2})$ be a random walk on $\mathcal{S}$: if $X_n = x \notin \{(1,1), (d,d)\}$ then X_{n+1} has the equal probability of being at one of the neighbors of x. Make both $(1,1)$ and (d,d) absorbing states. For example, if $x = (x_1, x_2)$ with $0 < x_i < d$ for $i = 1, 2$ then $P(x, y) = 1/4$ for each y in the set

$$\{(x_1, x_2 + 1),\ (x_1, x_2 - 1),\ (x_1 + 1, x_2),\ (x_1 - 1, x_2)\}$$

For x on the edges of the lattice, there are fewer neighbors and the transition probabilities will be slightly different.

Define $\tau = \inf\{n \in \mathbb{N} : X_n = (1,1) \text{ or } X_n = (d,d)\}$.
Define $B = \{\tau < \infty,\ X_\tau = (d,d)\}$.
Define $Z_n = X_{n,1} + X_{n,2}$ for each n.

(i) Write down the transition probabilities $P(x,y)$ when x is on the edge of the lattice. (That is, at least one of x_1 and x_2 is equal to 1 or d.)

(ii) Define $f(x) = x_1 + x_2$ for $x = (x_1, x_2) \in \mathcal{S}$. Show that f is harmonic. That is, show that $\mathbb{E}\left(f(X_1) \mid X_0 = x\right) = f(x)$ for each x.

(iii) Explain why $\mathbb{P}_x\{\tau < \infty\} = 1$ for every x in $\mathcal{S}$.

(iv) Show that Z_n is a martingale for each initial state x of the X_n-chain.

(v) Show that Z_n is also a Markov chain. Write down the state space and transition probabilities for the Z_n-chain.

(vi) Define $g(x) = \mathbb{P}_x B$. Show that g is a harmonic function.

(vii) Find $g(x)$ for each x in $\mathcal{S}$.

[3] Question 1 on Homework 3 described a modification of the queueing example from Section 2.3 of the Chang notes. I got myself greatly confused over the problem of independence of X_n and D_1, D_2,…, D_n. The following questions revisit the modified problem, with the aim of proving that

<1> $$\mathbb{P}_\pi\{X_2 = k, D_1 = \delta_1, D_2 = \delta_2\} = \pi_k \theta(\delta_1)\theta(\delta_2) \qquad \text{for all } \delta_i \in \{0,1\}, \text{ all } k \geq 0,$$

where $\theta(1) = p$ and $\theta(0) = 1-p$. That is, the aim is to prove independence of X_2, D_1, and D_2. To this end define

$$G(k, \delta_1, \alpha_1, \delta_2, \alpha_2) = \mathbb{P}_\pi\{X_2 = k, D_1 = \delta_1, A_1 = \alpha_1, D_2 = \delta_2, A_2 = \alpha_2\}.$$

and let $j = k - \alpha_2 + \delta_2$ and $i = j - \alpha_1 + \delta_1$. Note that δ_1 and α_1 are uniquely determined by i and j if $|i-j| = 1$; and δ_2 and α_2 are uniquely determined by j and k if $|j-k| = 1$

You may assume that the chain has stationary distribution π and transition probabilities as shown on the Solutions to Sheet 3.

(i) For each $i \geq 0$ define $f_i(\delta) = \mathbb{P}_i\{A_1 = D_1 = \delta\}$ for $\delta \in \{0,1\}$ and $i = 0, 1, 2, \ldots$. Write down the expression for $f_i(\delta)$. Hint: You will need to distinguish between the cases $i = 0$ and $i \geq 1$.

(ii) Explain why

$$\begin{aligned} &G(k,\delta_1, \alpha_1, \delta_2, \alpha_2) \\ &\qquad = \mathbb{P}_\pi\{X_0 = i, X_1 = j, X_2 = k, D_1 = \delta_1, A_1 = \alpha_1, D_2 = \delta_2, A_2 = \alpha_2\} \end{aligned}$$

for all $k \geq 1$ and all $\delta_i, \alpha_i \in \{0,1\}$.

(iii) If $\alpha_1 \neq \delta_1$ and $\alpha_2 \neq \delta_2$ show that

$$\begin{aligned} &G(k,\delta_1, \alpha_1, \delta_2, \alpha_2) \\ &\qquad = \mathbb{P}_\pi\{X_0 = k, X_1 = j, X_2 = i\} \\ &\qquad = \mathbb{P}_\pi\{X_0 = k, D_1 = \alpha_2, A_1 = \delta_2, D_2 = \alpha_1, A_2 = \delta_1\}. \end{aligned}$$

(iv) If $\alpha_1 = \delta_1$ and $\alpha_2 \neq \delta_2$ show that

$$\begin{aligned} G(k, &\delta_1, \alpha_1, \delta_2, \alpha_2) \\ &= \mathbb{P}_\pi\{X_2 = k, X_1 = j, X_0 = j, D_1 = \delta_1 = A_1\} \\ &= \pi_j f_j(\delta_1) P(j,k) \\ &= \mathbb{P}_\pi\{X_0 = k, D_1 = \alpha_2, A_1 = \delta_2, D_2 = \alpha_1, A_2 = \delta_1\}. \end{aligned}$$

(v) Similarly, if $\alpha_1 \neq \delta_1$ and $\alpha_2 = \delta_2$, show that

$$\begin{aligned} G(k, &\delta_1, \alpha_1, \delta_2, \alpha_2) \\ &= \mathbb{P}_\pi\{X_0 = k, D_1 = \alpha_2, A_1 = \delta_2, D_2 = \alpha_1, A_2 = \delta_1\}. \end{aligned}$$

(vi) If $\alpha_1 = \delta_1$ and $\alpha_2 = \delta_2$, show that

$$\begin{aligned} G(k, &\delta_1, \alpha_1, \delta_2, \alpha_2) \\ &= \mathbb{P}_\pi\{X_2 = k, X_1 = k, X_0 = k, D_1 = \delta_1 = A_1, D_2 = \delta_2 = A_2\} \\ &= \pi_k f_k(\delta_1) f_k(\delta_2) \\ &= \mathbb{P}_\pi\{X_0 = k, D_1 = \alpha_2, A_1 = \delta_2, D_2 = \alpha_1, A_2 = \delta_1\}. \end{aligned}$$

(vii) Complete the proof of <1>.

Bonus points if you find and correct any errors in this question.