Statistics 251b/551b, spring 2009
Take-home test 2
Due: Friday 24 April

- *You should prepare solutions to the following questions without help or hints from anybody else. In particular, if you have been working in a group you must suspend the group arrangement for the test; you must not discuss the questions with your group buddy.*
- *If you have questions of interpretation, or of clarification of the meaning of a question, ask David Pollard.*
- *Make sure you explain your calculations and notation.*
- *Each part of each question is worth 5 points.*
- *As usual, even if you are unable to solve one part of a question you may still use the result for the following parts.*

[1] An urn initially contains $X_0 = 2$ white balls and 1 black ball. Repeat: select a ball at random from the urn then return it plus one more ball of the same color to the urn. Let W_i equal 1 if the ith ball selected is white and 0 otherwise. Define $S_n = \sum_{i=1}^n W_i$. After n such steps there are $N_n = n + 3$ balls in the urn, $X_n = S_n + 2$ white and $n - S_n + 1$ black.

Write $\mathcal{F}_n$ for the information that is learned by observing the values of the random variables $X_0, X_1, \ldots, X_n$. [The text would probably write $X_{1,n}$ instead of $\mathcal{F}_n$. See Chang page 188.]

(i) Show that $Z_n := X_n/N_n$ for $n = 0, 1, \ldots$ is a martingale.

By Chang Theorem 4.38, Z_n converges with probability one to some random limit Z.

(ii) For each sequence $\delta_1, \ldots, \delta_n$ with $\delta_i \in \{0, 1\}$ and $\sum_{i=1}^n \delta_i = k$, show that

$$\mathbb{P}\{W_1 = \delta_1, W_2 = \delta_2, \ldots, W_n = \delta_n\} = 2\frac{(k+1)!(n-k)!}{(n+2)!}.$$

Hint: Experiment with some examples like $n = 4$ for various δ_i's to see the pattern. Try to explain why a similar pattern should appear in the general case.

(iii) Find the distribution of X_n. Hint: How many ways are there to put k ones and $n-k$ zeros into n places?

(iv) Suppose h is a function on $[0, 1]$. Explain why

$$\mathbb{E}h(Z_n) = \int_0^1 g_n(s)\, ds$$

where

$$g_n(s) = \sum_{k=0}^n \frac{2(k+1)}{n+2} h\left(\frac{k+2}{n+3}\right) \mathbf{1}\{t_k < s \le t_{k+1}\} \qquad \text{for } t_k = k/(n+1).$$

(v) If h is continuous, show that $g_n(s) \to 2s\,h(s)$ as $n \to \infty$, for each fixed s in $[0,1]$.

(vi) Deduce that $\mathbb{E}h(Z) = \int_0^1 2s\,h(s)\,ds$ for each bounded, continuous function h on $[0,1]$. [[You may assume the analog of the Bounded Convergence Theorem, as stated on page 227 of the Chang notes, for integrals of functions over $[0,1]$.]]

(vii) By taking a sequence of h functions that converge pointwise to $\mathbf{1}\{0 \le s \le t\}$, deduce that $\mathbb{P}\{Z \le t\} = t^2$ for each t in $[0,1]$.

[2] Suppose $X_0, X_1, \dots, X_N$ are nonnegative random variables. [If you like, you could assume that each X_i is discrete.] Write $\mathcal{F}_i$ for the information obtained by observing $X_0, X_1, \dots, X_i$. Define new random variables $Z_N = X_N$ and, recursively,

$$Z_i = \max(X_i, \mathbb{E}(Z_{i+1} \mid \mathcal{F}_i) \qquad \text{for } i = N-1, N-2, \dots, 0.$$

Note that $Z_i \ge X_i$ for each i.

(i) Show that $Z_0, Z_1, \dots, Z_n$ is a nonnegative supermartingale.

(ii) Suppose $Y_0, Y_1, \dots, Y_N$ is another supermartingale for which $Y_i \ge X_i$ for each i. Show that $Y_i \ge Z_i$. Hint: Start with $i = N$ then work your way back, one step at a time.

(iii) Define $\tau = \min\{i : Z_i = X_i\}$. Explain why τ is a stopping time. [Note that $X_\tau = Z_\tau$.]

(iv) If σ is another stopping time for which $\sigma \ge \tau$, show that $\mathbb{E}X_\tau \ge \mathbb{E}X_\sigma$. Hint: Note that $X_\tau = Z_\tau$.

(v) Define $W_i = Z_{\tau \wedge i}$ for $i = 0, 1, \dots, N$. [Remember $\tau \wedge i = \min(\tau, i)$.] Explain why

$$W_{i+1} - W_i = (Z_{i+1} - Z_i)\mathbf{1}\{\tau \ge i+1\}$$

(vi) Show that $W_0, W_1, \dots, W_N$ is a martingale. Hint: What do you know about Z_i when $\tau \ge i+1$?

[3] Suppose $X_t = \int_0^1 H_s\,dB_s$ for $0 \le t \le 1$, for B a standard Brownian motion, where H is a process for which there exists a sequence of simple predictable processes H_n such that $\int_0^1 \mathbb{E}|H(s) - H_n(s)|^2\,ds \to 0$.

Define $A(t) = \int_0^t H(s)^2\,ds$ and $N_t = X(t)^2 - A(t)$ for $0 \le t \le 1$. Similarly, define $X_n(t) = \int_0^t H_n(s)\,dB_s$ and $A_n(t) = \int_0^t H_n(s)^2\,ds$ for $0 \le t \le 1$. Remember that $\mathbb{E}X_1^2 = \int_0^1 \mathbb{E}H_s^2\,ds < \infty$ and

$$\mathbb{E}|X_n(t) - X(t)|^2 = \int_0^t \mathbb{E}|H_n(s) - H(s)|^2\,ds \to 0.$$

In class I started to prove that N_t is a martingale. I had reduced the task to the following problem. For a fixed pair of times $t' > t$ define $\Delta X = X_{t'} - X_t$ and $\Delta A = A_{t'} - A_t$. Show that

<1> $$\mathbb{E}\left((\Delta X)^2 W\right) = \mathbb{E}\left((\Delta A)W\right)$$

for each (bounded) random variable W that depends only on $\mathcal{F}_t$ information. Without loss of generality you may asssume $0 \le W \le 1$.

(i) Write ΔX_n for $X_n(t') - X_n(t)$ and ΔA_n for $A_n(t') - A_n(t)$. Prove that $\mathbb{E}\left((\Delta X_n)^2 W\right) = \mathbb{E}\left((\Delta A_n)W\right)$. You may assume that

$$H_n(s,\omega) = \sum\nolimits_{i \le m} h_i(\omega)\mathbf{1}\{t_i < s \le t_{i+1}\}$$

for some grid $0 = t_0 < t_1 < \dots t_{m+1} = 1$ and, with no loss of generality, that $t = t_j$ and $t' = t_k$, for some $j < k$.

(ii) Prove that $\mathbb{E}|\Delta X_n - \Delta X|^2 \to 0$ as n tends to infinity. Hint: $(a-b)^2 \le 2a^2 + 2b^2$ for all real numbers a and b.

(iii) Deduce that

$$|\mathbb{E}(\Delta X_n)^2 W - \mathbb{E}(\Delta X)^2 W| \le \mathbb{E}|(\Delta X_n)^2 - (\Delta X)^2| \to 0.$$

Hint: Look at one of the useful inequalities at the end of the sheet.

(iv) Show that

$$|\mathbb{E}A_n(t)W - \mathbb{E}A(t)W| \le \int_0^1 \mathbb{E}|H_n(s)^2 - H(s)^2|\,ds \to 0$$

Hint: Look at the other useful inequality at the end of the sheet.

(v) Complete the proof of equality <1>.

Some useful inequalities.

Suppose B and D are random variables and $A = B + D$. Then

$$\begin{aligned} |\mathbb{E}A^2 - \mathbb{E}B^2| &\le \mathbb{E}|A^2 - B^2| \\ &\le 2\mathbb{E}|BD| + \mathbb{E}D^2 \\ &\le 2\sqrt{\mathbb{E}B^2\mathbb{E}D^2} + \mathbb{E}D^2 \qquad \text{by Cauchy-Schwarz.} \end{aligned}$$

Similarly, if B_s, D_s and $A_s = B_s + D_s$ are processes indexed by s in $[0,1]$ then

$$\begin{aligned} &\int_0^1 \mathbb{E}|A_s^2 - B_s^2|\,ds \\ &\quad\le 2\int_0^1 \sqrt{\mathbb{E}B_s^2\mathbb{E}D_s^2}\,ds + \int_0^1 \mathbb{E}D_s^2\,ds \qquad \text{by Cauchy-Schwarz} \\ &\quad\le 2\sqrt{\int_0^1 \mathbb{E}B_s^2\,ds \int_0^1 \mathbb{E}D_s^2\,ds} + \int_0^1 \mathbb{E}D_s^2\,ds \qquad \text{by Cauchy-Schwarz.} \end{aligned}$$