Consider an irreducible Markov chain with a finite state space $S$. Show that every state $i$ is positive recurrent by following these steps. Write $S \setminus i$ for $\{j \in S : j \neq i\}$.

(i) [5 points] Explain why, for each $j$ in $S \setminus i$, there exists a positive integer $N(j)$ such that $P_i\{X_{N(j)} = i\} > 0$. Define $N := 1 + \max_j N(j)$. Deduce that the quantity $\delta$ defined by

$$\delta := \min_{j \in S \setminus i} P_j\{T_i \leq N - 1\}$$

is strictly positive and that $P_j\{T_i \geq N\} \leq 1 - \delta$ for all $j$ in $S \setminus i$.

(ii) [5 points] For all $j$ and $k$ in $S \setminus i$, explain why

$$P_j\{T_i \geq 2N \mid X_N = k, T_i \geq N\} = P_k\{T_i \geq N\}$$

Hint: $\{T_i \geq m\} = \cap_{n=1}^{m-1}\{X_n \neq i\}$.

(iii) [5 points] Deduce that $P_j\{T_i \geq 2N\} \leq (1 - \delta)^2$ for all $j$ in $S \setminus i$.

(iv) [bonus points] Show that $P_j\{T_i \geq mN\} \leq (1 - \delta)^m$ for all $m \in \mathbb{N}$ and all $j$ in $S \setminus i$.

(v) [5 points] Deduce that $P_i\{T_i \geq mN + 1\} \leq (1 - \delta)^m$ for each $m$ in $\mathbb{N}$.

(vi) [5 points] Explain why

$$T_i \leq N + N \sum_{m \in \mathbb{N}} 1\{T_i \geq mN + 1\}.$$ 

Hint: Consider what happens to the right-hand side when $N < T_i \leq 2N$.

(vii) [5 points] Conclude that $\mathbb{E}_iT_i < \infty$. That is, $i$ is a positive recurrent state.

[2] [10 points] Consider a Markov chain with state space $S = \{0, 1, 2, \ldots\}$ and transition probabilities

$$\alpha = P(i, i + 1) \quad \text{for all } i \geq 0$$

$$\beta = P(0, 0) = P(i, i - 1) \quad \text{for all } i \geq 1$$

where $\alpha + \beta = 1$ and $\alpha > 1/2$. Suppose we generate the chain by repeated tossing of a coin that lands heads with probability $\alpha$, moving one step to the right for a head and one step to the left (or just stay where we are if $X_n = 0$) for a tail. Let $S_n$ denote the number of tails in the first $n$ tosses.

(i) [5 points] Suppose the chain starts in state 0. Explain why $X_n \geq (n - S_n) - S_n$ for all $n$.

(ii) [5 points] Show that $P_0\{X_n = 0\} \leq P\{S_n \geq n(1 + 2\delta)\}$ where $\delta = (\alpha - \beta)/4\beta$.

(iii) [5 points] Use the Fact about the Binomial distribution (see next page) to deduce that the chain is transient when $\alpha > 1/2$. 
FACT (ABOUT THE BINOMIAL DISTRIBUTION) Suppose $Y \sim \text{Bin}(n, \theta)$ for some $0 < \theta < 1$. Then for each $\delta > 0$,

$$\mathbb{P}\{Y \geq n\theta(1 + 2\delta)\} \leq \rho^{-n} \quad \text{where} \quad \rho = (1 + \delta)^{\delta \theta}.$$ 

PROOF For each $s > 0$

$$1\{Y \geq n\theta(1 + 2\delta)\} \leq (1 + s)^{Y - n\theta(1 + 2\delta)}$$

because the right-hand side is always nonnegative and it is at least 1 when $Y \geq n\theta(1 + 2\delta)$. Take expectations, using the fact that $\mathbb{E} (1 + s)^Y = (1 + \theta s)^n$, to deduce that

$$\mathbb{P}\{Y \geq n\theta(1 + 2\delta)\} \leq \left( \frac{1 + 2s\theta}{(1 + s)^{\theta + \theta\delta}} \right)^n \left( \frac{1}{(1 + s)^{\theta\delta}} \right)^n$$

Choose $s$ equal to $\delta$, which makes the second term equal to $\rho^{-n}$. The logarithm of the first term then equals $n$ times

$$\log(1 + \theta\delta) - \theta(1 + \delta) \log(1 + \delta).$$

Invoke the inequality

$$\frac{x}{1 + x} \leq \log(1 + x) \leq x \quad \text{for} \quad x > 0$$

to bound the last difference from above by $\theta\delta - \theta(1 + \delta)\delta/(1 + \delta) = 0$. 

$\square$