Statistics 251b/551b, spring 2009
Homework # 1
Due: Monday 26 January

If you are not able to solve a part of a problem, you can still get credit for later parts: Just assume the truth of what you unable to prove in the earlier part.

[1] Consider an irreducible Markov chain with a finite state space $\mathcal{S}$. Show that every state i is positive recurrent by following these steps. Write $\mathcal{S}\backslash i$ for $\{j \in \mathcal{S} : j \neq i\}$.

(i) [5 points] Explain why, for each j in $\mathcal{S}\backslash i$, there exists a positive integer $N(j)$ such $\mathbb{P}_i\{X_{N(j)} = i\} > 0$. Define $N := 1 + \max_j N(j)$. Deduce that the quantity δ defined by

$$\delta := \min_{j\in\mathcal{S}\backslash i} \mathbb{P}_j\{T_i \leq N-1\}$$

is strictly positive and that $\mathbb{P}_j\{T_i \geq N\} \leq 1-\delta$ for all j in $\mathcal{S}\backslash i$.

(ii) [5 points] For all j and k in $\mathcal{S}\backslash i$, explain why

$$\mathbb{P}_j\{T_i \geq 2N \mid X_N = k,\, T_i \geq N\} = \mathbb{P}_k\{T_i \geq N\}$$

Hint: $\{T_i \geq m\} = \cap_{n=1}^{m-1}\{X_n \neq i\}$.

(iii) [5 points] Deduce that $\mathbb{P}_j\{T_i \geq 2N\} \leq (1-\delta)^2$ for all j in $\mathcal{S}\backslash i$.

(iv) [bonus points] Show that $\mathbb{P}_j\{T_i \geq mN\} \leq (1-\delta)^m$ for all $m \in \mathbb{N}$ and all $j \in \mathcal{S}\backslash i$.

(v) [5 points] Deduce that $\mathbb{P}_i\{T_i \geq mN+1\} \leq (1-\delta)^m$ for each m in $\mathbb{N}$.

(vi) [5 points] Explain why

$$T_i \leq N + N\sum\nolimits_{m\in\mathbb{N}} \mathbf{1}\{T_i \geq mN+1\}.$$

Hint: Consider what happens to the right-hand side when $N < T_i \leq 2N$.

(vii) [5 points] Conclude that $\mathbb{E}_i T_i < \infty$. That is, i is a positive recurrent state.

[2] [10 points] Consider a Markov chain with state space $\mathcal{S} = \{0, 1, 2, \ldots\}$ and transition probabilities

$$\alpha = P(i, i+1) \qquad \text{for all } i \geq 0$$
$$\beta = P(0,0) = P(i, i-1) \qquad \text{for all } i \geq 1$$

where $\alpha+\beta = 1$ and $\alpha > 1/2$. Suppose we generate the chain by repeated tossing of a coin that lands heads with probability α, moving one step to the right for a head and one step to the left (or just stay where we are if $X_n = 0$) for a tail. Let S_n denote the number of tails in the first n tosses.

(i) [5 points] Suppose the chain starts in state 0. Explain why $X_n \geq (n - S_n) - S_n$ for all n.

(ii) [5 points] Show that $\mathbb{P}_0\{X_n = 0\} \leq \mathbb{P}\{S_n \geq n\beta(1+2\delta)\}$ where $\delta = (\alpha-\beta)/4\beta$.

(iii) [5 points] Use the Fact about the Binomial distribution (see next page) to deduce that the chain is transient when $\alpha > 1/2$.

FACT (ABOUT THE BINOMIAL DISTRIBUTION) Suppose $Y \sim \text{Bin}(n, \theta)$ for some $0 < \theta < 1$. Then for each $\delta > 0$,

$$\mathbb{P}\{Y \geq n\theta(1+2\delta)\} \leq \rho^{-n} \qquad \text{where } \rho = (1+\delta)^{\delta\theta}.$$

PROOF For each $s > 0$

$$\mathbf{1}\{Y \geq n\theta(1+2\delta)\} \leq (1+s)^{Y-n\theta(1+2\delta)}$$

because the right-hand side is always nonnegative and it is at least 1 when $Y \geq n\theta(1+2\delta)$. Take expectations, using the fact that $\mathbb{E}(1+s)^Y = (1+\theta s)^n$, to deduce that

$$\mathbb{P}\{Y \geq n\theta(1+2\delta)\} \leq \left(\frac{1+2s\theta}{(1+s)^{\theta+\theta\delta}}\right)^n \left(\frac{1}{(1+s)^{\theta\delta}}\right)^n$$

Choose s equal to δ, which makes the second term equal to ρ^{-n}. The logarithm of the first term then equals n times

$$\log(1+\theta\delta) - \theta(1+\delta)\log(1+\delta).$$

Invoke the inequality

$$\frac{x}{1+x} \leq \log(1+x) \leq x \qquad \text{for } x > 0$$

to bound the last difference from above by $\theta\delta - \theta(1+\delta)\delta/(1+\delta) = 0$.

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