Statistics 251b/551b, spring 2009
Homework # 2
Due: Monday 2 February

[1] Consider an irreducible, positive recurrent Markov chain $\{X_n : n = 0, 1, 2, \dots\}$ with state space $\mathcal{S}$ and transition probabilities $P(i, j)$. Suppose the chain has period 2. Let α be some arbitrarily chosen state, which will stay fixed throughout the problem. You know that there exists a stationary probability distribution π for which $\pi_\alpha = 1/\mathbb{E}_\alpha T_\alpha$.

(i) [5 points] For each j in $\mathcal{S}$ define $\mathbb{N}_j = \{n \in \mathbb{N} : \mathbb{P}_\alpha\{X_n = j\} > 0\}$. Explain why the elements of $\mathbb{N}_j$ are either all odd or all even. Hint: If $n_1, n_2 \in \mathbb{N}_j$ and $\mathbb{P}_j\{X_\ell = i\} > 0$, what do you know about $n_1 + \ell$ and $n_2 + \ell$?

(ii) [5 points] Define $\mathcal{S}_{\text{even}}$ to be the the set of states j for which all elements of $\mathbb{N}_j$ are even and $\mathcal{S}_{\text{odd}}$ for the remaining states. Show that $\pi(\mathcal{S}_{\text{odd}}) = \pi(\mathcal{S}_{\text{even}}) = 1/2$. Hint: If $j \in \mathcal{S}_{\text{odd}}$ for which i is it possible to have $P(i, j) > 0$?

(iii) [5 points] Define two probability distributions: $\lambda_i = 2\pi_i$ if $i \in \mathcal{S}_{\text{even}}$ and $\lambda_i = 0$ otherwise; and $\mu_i = 2\pi_i$ if $i \in \mathcal{S}_{\text{odd}}$ and $\mu_i = 0$ otherwise. Show that $\lambda P = \mu$ and $\mu P = \lambda$.

(iv) [5 points] Define $\widetilde{X}_n = X_{2n}$ for $n = 0, 1, 2, \dots$. Explain why $\widetilde{X}_n$ has transition probability matrix P^2. Explain why, for the P^2 chain, all states in $\mathcal{S}_{\text{even}}$ communicate but no state in $\mathcal{S}_{\text{odd}}$ is accessible from a state in $\mathcal{S}_{\text{even}}$.

(v) [5 points] Explain why the P^2 chain is aperiodic.

(vi) [10 points] For each initial distribution ν that concentrates on $\mathcal{S}_{\text{even}}$, show that

$$\sum_{i \in \mathcal{S}_{\text{even}}} |\mathbb{P}_\nu\{X_{2n} = i\} - \lambda_i| \to 0 \qquad \text{as } n \to \infty.$$

Hint: You may invoke the Basic Limit Theorem as stated in class or in the text.

(vii) [5 points] For each initial distribution ν that concentrates on $\mathcal{S}_{\text{even}}$, show that

$$\sum_{i \in \mathcal{S}_{\text{odd}}} |\mathbb{P}_\nu\{X_{2n+1} = i\} - \mu_i| \to 0 \qquad \text{as } n \to \infty.$$

(viii) [bonus points] For an arbitrary initial distribution ν on $\mathcal{S}$, describe the behavior of $\mathbb{P}_\nu\{X_n = i\}$ as n tends to infinity. In particular, discuss whether there is a unique stationary probability distribution for the P-chain.

[2] [20 points] Chang notes, Problem 1.17. Some hints will be given in class.