Statistics 251b/551b, spring 2009
Homework #4
Due: Monday 16 February

For each real number $x$, the positive part $x^+$ is defined as $\max(0, x)$ and the negative part is defined as $\max(0, -x)$. Note that $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

[1] Suppose $\lambda$ and $\mu$ are probability measures on a countable set $\mathcal{S}$. The total variation distance $\|\lambda - \mu\|_{\text{TV}}$ is defined as $\sup_{A\subseteq\mathcal{S}} |\lambda A - \mu A|$.

(i) [10 points] Show that the supremum in the definition must be achieved either by the set $A_0 = \{i \in \mathcal{S} : \lambda_i \geq \mu_i\}$ or by the set $A_1 = \{i \in \mathcal{S} : \lambda_i \leq \mu_i\}$.

(ii) [10 points] Deduce that $\|\lambda - \mu\|_{\text{TV}} = \max(\alpha_0, \alpha_1)$ where $\alpha_0 = \sum_{i\in\mathcal{S}}(\lambda_i - \mu_i)^+$ and $\alpha_1 = \sum_{i\in\mathcal{S}}(\lambda_i - \mu_i)^-$.

(iii) [10 points] Show that $\alpha_0 = \alpha_1 = \frac{1}{2}\sum_{i\in\mathcal{S}} |\lambda_i - \mu_i|$.

[2] Let $\mathcal{G}$ be a finite, connected graph with vertex set $\mathcal{S}$ and edge set $\mathcal{E}$. For each edge $e$ suppose $w_e$ is a strictly positive weight. Define $W_i = \sum_{\{i,j\}\in\mathcal{E}} w_{ij}$. The random walk on the weighted graph has transition probabilities

$$Q(i, j) = w_{i,j}/W_i \qquad \text{if } \{i, j\} \in \mathcal{E}.$$

Suppose $\lambda$ is a probability distribution on $\mathcal{S}$ for which $\max_{i\in\mathcal{S}} \lambda_i/W_i > \min_{i\in\mathcal{S}} \lambda_i/W_i$.

(i) [10 points] Explain why there must exist at least one edge $\{i, j\}$ for which $\lambda_i/W_i > \lambda_j/W_j$.

(ii) [30 points] Explain why the the chain with transition probabilities

$$P(i, j) = Q(i, j) \min\left(1, \frac{\lambda_j Q(j, i)}{\lambda_i Q(i, j)}\right) \qquad \text{for } \{i, j\} \in \mathcal{E}$$

is irreducible, aperiodic, and positive recurrent.

[3] [10+10+10 points] Chang Problem 2.20.

[4] [many bonus points] Chang Problem 2.21. Rewards for any (very) intelligent discussion or the requested example.