For the first two questions, \( A_\theta \) denotes the matrix \( \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \) for an angle \( \theta \).

[1] Suppose \( W \) is a random vector that is uniformly distributed on the set \( \{ w \in \mathbb{R}^2 : |w| = 1 \} \). Let \( \mathbb{E}W = \mu \), a vector with components \( \mu_1 \) and \( \mu_2 \), and \( \text{var}(W) = S \).

(i) [5 points] Explain why \( A_\theta W \) has the same distribution as \( W \), for each fixed \( \theta \).

(ii) [5 points] Explain why \( \mathbb{E}(A_\theta W) = A_\theta \mu \) and \( \text{var}(A_\theta W) = A_\theta \text{var}(W) A_\theta' \).

(iii) [10 points] By an appropriate choice of \( \theta \), deduce that \( \mu = 0 \) and \( S = \frac{1}{2} I_2 \), where \( I_2 \) is the \( 2 \times 2 \) identity matrix.

[2] Let \( \{ W_t : t \geq 0 \} \) be a two-dimensional Brownian motion, that is, its components \( \{ X_t : t \geq 0 \} \) and \( \{ Y_t : t \geq 0 \} \) are independent standard Brownian motions. Explain why \( \{ A_\theta W_t : t \geq 0 \} \), for a fixed \( \theta \), is also a two-dimensional Brownian motion.

[3] Suppose \( \{ M_t : 0 \leq t \leq 1 \} \) is a martingale with continuous sample paths. That is, \( M_t \) is determined by the information, \( \mathcal{F}_t \), available at time \( t \) and \( \mathbb{E}(M_t - M_s \mid \mathcal{F}_s) = 0 \) for each pair of times \( 0 \leq s < t \leq 1 \). Equivalently, \( \mathbb{E}((M_t - M_s)W) = 0 \) for each random variable \( W \) that depends only on the information \( \mathcal{F}_s \).

Suppose also that there is some process \( \{ A_t : 0 \leq t \leq 1 \} \), with continuous sample paths \( A(t, \omega) \) that are increasing in \( t \), such that \( N_t = M_t^2 - A_t \) is a martingale. (Necessarily, \( A_t \) is determined by the \( \mathcal{F}_t \) information.) For simplicity, assume \( M_0 = A_0 = 0 \).

(i) [10 points] Let \( G \) be a grid of time points \( 0 = t_0 < t_1 < \cdots < t_n \leq 1 \). Write \( \Delta_i M \) for \( M(t_{i+1}, \omega) - M(t_i, \omega) \). Define \( \Delta_i N \) and \( \Delta_i A \) similarly. Show that

\[
\mathbb{E}((\Delta_i M)^2 - \Delta_i A \mid \mathcal{F}_{t_i}) = 0 \quad \text{for } i = 0, 1, \ldots, n - 1.
\]

Hint: What do you know about \( \mathbb{E}(\Delta_i N \mid \mathcal{F}_{t_i}) \)?

(ii) [10 points] Suppose \( H_G \) is a simple process, that is,

\[
H_G(s, \omega) = \sum_{0 \leq i < n} h_i(\omega) \mathbf{1}\{ t_i < s \leq t_{i+1} \}
\]

where \( h_i \) is a random variable that depends only on \( \mathcal{F}_{t_i} \) information. (It will behave like a constant when you take expectations conditional on \( \mathcal{F}_{t_i} \).) Define

\[
Y_G(1) = \int_0^1 H_G(s) dM(s) = \sum_{0 \leq i < n} h_i(\omega) \Delta_i M.
\]
Show that $E[Y_G(1)] = 0$ and $E[Y_G^2(1)] = E\left(\int_0^1 H_G^2(s, \omega) dA(s)\right)$. [[Remember that $\int_0^1 f(s) dA(s) = \sum_i f_i \Delta_i A$ if $f = \sum_{0 \leq i < n} f_i 1\{t_i < s \leq t_{i+1}\}$. ]] Hint: Part (i) should help with terms like $E[h_i^2(\Delta_i M)^2]$.

(iii) [10 points] For each fixed $t$ in $[0, 1]$, show that

$$H_G(s, \omega) 1\{0 < s \leq t\} = \sum_{0 \leq i < k} h_i(\omega) 1\{t_i < s \leq t_{i+1}\}$$

$$+ h_k 1\{t_k < s \leq t\} \quad \text{if } t_k < t \leq t_{k+1}$$

$$= \sum_{0 \leq i < n} h_i(\omega) 1\{t_i \wedge t < s \leq t_{i+1} \wedge t\},$$

a simple process defined for a slightly different grid of points.

(iv) [10 points] Define

$$Y_G(t) = \int_0^t H_G(s) 1\{0 < s \leq t\} dM(s) \quad \text{for } 0 \leq t \leq 1.$$ 

Show that $\{Y_G(t) : 0 \leq t \leq 1\}$ has continuous sample paths. Hint: Use the second form in part (iii) for the integrand.

(v) [10 points] Show that $\{Y_G(t) : 0 \leq t \leq 1\}$ is a martingale. Hint: If $s < t$, you may suppose $s = t_j$ and $t = t_k$ for some $j < k$. (If $s$ and $t$ were not grid points, you could refine the grid by adding them in. None of the preceding definitions would be significantly changed.)