Statistics 251b/551b, spring 2009
Homework #7
Due: Monday 6 April

For the first two questions, A_θ denotes the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ for an angle θ.

[1] Suppose W is a random vector that is uniformly distributed on the set $\{w \in \mathbb{R}^2 : |w| = 1\}$. Let $\mathbb{E}W = \mu$, a vector with components μ_1 and μ_2, and var$(W) = S$.

(i) [5 points] Explain why $A_\theta W$ has the same distribution as W, for each fixed θ.

(ii) [5 points] Explain why $\mathbb{E}(A_\theta W) = A_\theta \mu$ and $\text{var}(A_\theta W) = A_\theta \text{var}(W) A'_\theta$.

(iii) [10 points] By an appropriate choice of θ, deduce that $\mu = 0$ and $S = \frac{1}{2} I_2$, where I_2 is the 2×2 identity matrix.

[2] [20 points] Let $\{W_t : t \geq 0\}$ be a two-dimensional Brownian motion, that is, its components $\{X_t : t \geq 0\}$ and $\{Y_t : t \geq 0\}$ are independent standard Brownian motions. Explain why $\{A_\theta W_t : t \geq 0\}$, for a fixed θ, is also a two-dimensional Brownian motion.

[3] Suppose $\{M_t : 0 \leq t \leq 1\}$ is a martingale with continuous sample paths. That is, M_t is determined by the information, $\mathcal{F}_t$, available at time t and $\mathbb{E}\left(M_t - M_s \mid \mathcal{F}_s\right) = 0$ for each pair of times $0 \leq s < t \leq 1$. Equivalently, $\mathbb{E}\left((M_t - M_s)W\right) = 0$ for each random variable W that depends only on the information $\mathcal{F}_s$.

Suppose also that there is some process $\{A_t : 0 \leq t \leq 1\}$, with continuous sample paths $A(t, \omega)$ that are increasing in t, such that $N_t = M_t^2 - A_t$ is a martingale. (Necessarily, A_t is determined by the $\mathcal{F}_t$ information.) For simplicity, assume $M_0 = A_0 = 0$.

(i) [10 points] Let G be a grid of time points $0 = t_0 < t_1 < \cdots < t_n \leq 1$. Write $\Delta_i M$ for $M(t_{i+1}, \omega) - M(t_i, \omega)$. Define $\Delta_i N$ and $\Delta_i A$ similarly. Show that

$$\mathbb{E}\left((\Delta_i M)^2 - \Delta_i A \mid \mathcal{F}_{t_i}\right) = 0 \qquad \text{for } i = 0, 1, \ldots, n-1.$$

Hint: What do you know about $\mathbb{E}\left(\Delta_i N \mid \mathcal{F}_{t_i}\right)$?

(ii) [10 points] Suppose H_G is a simple process, that is,

$$H_G(s, \omega) = \sum\nolimits_{0 \leq i < n} h_i(\omega) \mathbf{1}\{t_i < s \leq t_{i+i}\}$$

where h_i is a random variable that depends only on $\mathcal{F}_{t_i}$ information. (It will behave like a constant when you take expectations conditional on $\mathcal{F}_{t_i}$.) Define

$$Y_G(1) = \int_0^1 H_G(s) dM(s) = \sum\nolimits_{0 \leq i < n} h_i(\omega) \Delta_i M.$$

Show that $\mathbb{E}Y_G(1) = 0$ and $\mathbb{E}Y_G^2(1) = \mathbb{E}\left(\int_0^1 H_G^2(s,\omega)dA(s)\right)$. [[Remember that $\int_0^1 f(s)dA(s) = \sum_i f_i \Delta_i A$ if $f = \sum_{0\le i<n} f_i \mathbf{1}\{t_i < s \le t_{i+1}\}$.]] Hint: Part (i) should help with terms like $\mathbb{E}h_i^2(\Delta_i M)^2$.

(iii) [10 points] For each fixed t in $[0,1]$, show that

$$\begin{aligned} H_G(s,\omega)\mathbf{1}\{0 < s \le t\} &= \sum\nolimits_{0\le i<k} h_i(\omega)\mathbf{1}\{t_i < s \le t_{i+1}\} \\ &\quad + h_k \mathbf{1}\{t_k < s \le t\} \qquad \text{if } t_k < t \le t_{k+1} \\ &= \sum\nolimits_{0\le i<n} h_i(\omega)\mathbf{1}\{t_i \wedge t < s \le t_{i+1} \wedge t\}, \end{aligned}$$

a simple process defined for a slightly different grid of points.

(iv) [10 points] Define

$$Y_G(t) = \int_0^t H_G(s)\mathbf{1}\{0 < s \le t\}\, dM(s) \qquad \text{for } 0 \le t \le 1.$$

Show that $\{Y_G(t) : 0 \le t \le 1\}$ has continuous sample paths. Hint: Use the second form in part (iii) for the integrand.

(v) [10 points] Show that $\{Y_G(t) : 0 \le t \le 1\}$ is a martingale. Hint: If $s < t$, you may suppose $s = t_j$ and $t = t_k$ for some $j < k$. (If s and t were not grid points, you could refine the grid by adding them in. None of the preceding definitions would be significantly changed.)