Statistics 251b/551b, spring 2009
Homework # 1 solutions

[1] *Consider an irreducible Markov chain with a finite state space $\mathcal{S}$. Show that every state i is positive recurrent by following these steps. Write $\mathcal{S}\backslash i$ for $\{j \in \mathcal{S} : j \neq i\}$.*

Most of the difficulties students had with this problem came from an improper application of the Markov property or faulty conditioning. I hope it will help if I give a more detailed treatment of these ideas than appears in the Chang notes.

The Markov property can be stated more formally as: for every initial distribution μ, for every event D that involves only statements about $X_0, X_1, \ldots, X_{n-1}$, and every function f,

<1> $$\mathbb{E}_\mu \left(f(X_n, X_{n+1}, X_{n+2}, \ldots) \mid X_n = i,\ D \right) = \mathbb{E}_i f(X_0, X_1, X_2, \ldots).$$

In particular, when $f(X_n, X_{n+1}, X_{n+2}, \ldots) = \mathbf{1}\{X_{n+1} = j\}$,

<2> $$\mathbb{P}_\mu\{X_{n+1} = j \mid X_n = i,\ D\} = \mathbb{P}_i\{X_1 = j\} = P(i,j),$$

which is the definition given in the Chang notes. A formal proof of <1> would involve repeated conditioning arguments (using <2>) involving the successive states X_{n+1}, X_{n+2}, and so on.

Now consider the time $T_i = \inf\{n \in \mathbb{N} : X_n = i\}$. For fixed $m, M \in \mathbb{N}$, the event $\{T_i \geq m + M\}$ can be written as $\{X_n \neq i : \text{ for } 1 \leq n < m + M\}$. The indicator function of that event can be written as a product of random variables, GHF, where

$$G = \mathbf{1}\{X_n \neq i \text{ for } 1 \leq n < m\}$$
$$H = \mathbf{1}\{X_m \neq i\}$$
$$F = f(X_{m+1}, \ldots, X_{m+M-1})$$

for the function

$$f(y_{m+1}, \ldots, y_{m+M-1}) = \mathbf{1}\{y_n \neq i : \text{ for } m + 1 \leq n \leq m + M - 1\}.$$

Remark. Notice how I have written the function using dummy variables $y_{m+1}, \ldots, y_{m+M-1}$. It is important to think of f in this way if we want to replace $X_{m+1}, X_{m+2}, \ldots$ by $X_1, X_2, \ldots$.

Thus

$$\mathbb{P}_\mu\{T_i \geq m+M\} = \mathbb{E}_\mu \mathbf{1}\{T_i \geq m+M\} = \mathbb{E}_\mu GHF.$$

For the next step, write D for the event $\{T_i \geq m\}$, so that $\mathbf{1}_D = G$. Split according to the events D or D^c, then further split D according to the values that can be taken by X_m. Condition.

$$\begin{aligned}
&\mathbb{P}_\mu\{T_i \geq m+M\} \\
&\quad = \mathbb{E}_\mu\left(GHF \mid D^c\right)\mathbb{P}_\mu D^c \\
&\qquad + \sum\nolimits_{k\in\mathcal{S}} \mathbb{E}_\mu\left(GHF \mid X_m = k,\, D\right)\mathbb{P}_\mu\left(\{X_m = k\}\cap D\right).
\end{aligned}$$

The first term on the right-hand side is zero, because occurence of the event D^c implies that $T_i < m$ and $G = 0$. Similarly, occurence of the event D implies that $G = 1$, which gives the simplification

$$\mathbb{E}_\mu\left(GHF \mid X_m = k,\, D\right) = \mathbb{E}_\mu\left(HF \mid X_m = k,\, D\right).$$

If $k = i$, occurence of the event $\{X_m = k\}$ implies $H = 0$; and occurence of the event $\{X_m = k\}$ for any $k \neq i$ implies $H = 1$. With these simplifications we get

$$\mathbb{P}_\mu\{T_i \geq m+M\} = \sum\nolimits_{k\in\mathcal{S}\backslash i} \mathbb{E}_\mu\left(F \mid X_m = k,\, D\right)\mathbb{P}_\mu\left(\{X_m = k\}\cap D\right).$$

Written out in full, the conditional expectation $\mathbb{E}_\mu\left(F \mid X_m = k,\, D\right)$ becomes

$$\mathbb{E}_\mu\left(f(X_{m+1},\dots,X_{m+M-1}) \mid X_m = k,\, X_n \neq i \text{ for } 1 \leq n < m\right)$$

which, by the Markov property <1>, equals

$$\mathbb{E}_k f(X_1, X_2, \dots, X_{M-1}) = \mathbb{P}_k\{X_n \neq i \text{ for } 1 \leq n \leq M-1\} = \mathbb{P}_k\{T_i \geq M\}.$$

In summary:

$$\mathbb{P}_\mu\{T_i \geq m+M\} = \sum\nolimits_{k\in\mathcal{S}\backslash i} \mathbb{P}_k\{T_i \geq M\}\mathbb{P}_\mu\{X_m = k,\, T_i \geq m\}$$

If we know that $\mathbb{P}_k\{T_i \geq M\} \leq C$ for all k in $\mathcal{S}\backslash i$, the final sum is bounded above by

$$C\sum\nolimits_{k\in\mathcal{S}\backslash i} \mathbb{P}_\mu\{X_m = k,\, T_i \geq m\} = C\mathbb{P}_\mu\{X_m \neq i,\, T_i \geq m\}.$$

That is,

$$\mathbb{P}_\mu\{T_i \geq m+M\} \leq C\mathbb{P}_\mu\{T_i \geq m+1\}.$$

To solve the homework problem, you needed to apply this inequality repeatedly, for various values of m and M.

[2] You were supposed to argue that

$$\begin{aligned} \mathbb{E}_0 N_0 &= \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_0\{X_n = 0\} \\ &\le \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_0\{S_n \ge n/2\} \\ &\le \sum\nolimits_{n\in\mathbb{N}} \rho^{-n} \qquad \text{for some } \rho > 1 \\ &< \infty \end{aligned}$$

thereby establishing transience of state 0 when $\alpha > 1/2$.