Statistics 251b/551b, spring 2009
Homework # 2 solutions

[1] *Consider an irreducible, positive recurrent Markov chain $\{X_n : n = 0, 1, 2, \dots\}$ with state space $\mathcal{S}$ and transition probabilities $P(i, j)$. Suppose the chain has period 2. Let α be some arbitrarily chosen state, which will stay fixed throughout the problem. You know that there exists a stationary probability distribution π for which $\pi_\alpha = 1/\mathbb{E}_\alpha T_\alpha$.*

To simplify the notation, write $i \xrightarrow{m} j$ to mean $\mathbb{P}_i\{X_m = j\} > 0$. Thus $\mathbb{N}_j = \{n \in \mathbb{N} : \alpha \xrightarrow{n} j\}$. You should convince yourself that (i) $P(i, j) > 0$ if and only if $i \xrightarrow{1} j$ and (ii) if $i \xrightarrow{m} j$ and $j \xrightarrow{n} k$ then $i \xrightarrow{m+n} k$.

(i) *Explain why the elements of $\mathbb{N}_j$ are either all odd or all even.*

If $\alpha \xrightarrow{n} j$ and $j \xrightarrow{\ell} \alpha$ then $\alpha \xrightarrow{n+\ell} \alpha$. By definition of periodicity, $n + \ell$ must be even, for every n in $\mathbb{N}_j$. If ℓ is odd, deduce that all n in $\mathbb{N}_j$ are odd; if ℓ is even, deduce that all n in $\mathbb{N}_j$ are even.

The stationary distribution is defined to satisfy the equation

$$\sum\nolimits_{i\in\mathcal{S}} \pi_i P(i, j) = \pi_j$$

If $P(i, j) > 0$ and $\alpha \xrightarrow{n} i$ then $\alpha \xrightarrow{n+1} j$. If $j \in \mathcal{S}_{\text{odd}}$ then $n + 1$ must be odd, implying $i \in \mathcal{S}_{\text{even}}$. Similarly, if $j \in \mathcal{S}_{\text{even}}$ then $i \in \mathcal{S}_{\text{odd}}$. The equations for π can be rewritten as

$$\sum\nolimits_{i\in\mathcal{S}_{\text{even}}} \pi_i P(i, j) = \pi_j \qquad \text{if } j \in \mathcal{S}_{\text{odd}} \qquad \text{<1>}$$

$$\sum\nolimits_{i\in\mathcal{S}_{\text{odd}}} \pi_i P(i, j) = \pi_j \qquad \text{if } j \in \mathcal{S}_{\text{even}} \qquad \text{<2>}$$

(ii) *Show that $\pi(\mathcal{S}_{\text{odd}}) = \pi(\mathcal{S}_{\text{even}}) = 1/2$.*

Sum equality <1> over all j in $\mathcal{S}_{\text{odd}}$ then interchange the order of summation on the left-hand side to get

$$\sum\nolimits_{i\in\mathcal{S}_{\text{even}}} \pi_i \sum\nolimits_{j\in\mathcal{S}_{\text{odd}}} P(i, j) = \pi(\mathcal{S}_{\text{odd}}).$$

The first expression reduces to $\pi(\mathcal{S}_{\text{even}})$ because $\sum_{j\in\mathcal{S}_{\text{odd}}} P(i, j) = 1$ when $i \in \mathcal{S}_{\text{even}}$.

(iii) *Define two probability distributions: $\lambda_i = 2\pi_i$ if $i \in \mathcal{S}_{\text{even}}$ and $\lambda_i = 0$ otherwise; and $\mu_i = 2\pi_i$ if $i \in \mathcal{S}_{\text{odd}}$ and $\mu_i = 0$ otherwise. Show that $\lambda P = \mu$ and $\mu P = \lambda$.*

Multiply both sides of <1> and <2> by 2.

(iv) *Define $\widetilde{X}_n = X_{2n}$ for $n = 0, 1, 2, \dots$. Explain why $\widetilde{X}_n$ has transition probability matrix P^2. Explain why, for the P^2 chain, all states in $\mathcal{S}_{\text{even}}$ communicate but no state in $\mathcal{S}_{\text{odd}}$ is accessible from a state in $\mathcal{S}_{\text{even}}$.*

See Chang notes §1.3 for the P^2 interpretation.

For each $i \in \mathbb{S}$ there exists some integers m and n for which $i \xrightarrow{m} \alpha$ and $\alpha \xrightarrow{n} i$. Note that $m+n$ must be even because state α has period 2. If $i \in \mathbb{S}_{\text{even}}$ then n is even, forcing m also to be even. That is, for each $i \in \mathbb{S}_{\text{even}}$ there is a path leading from i to α in an even number of steps and a path leading from a to i in an even number of steps. Under P^2, the set $\mathbb{S}_{\text{even}}$ is irreducible. The argument for $\mathbb{S}_{\text{odd}}$ is similar.

On the other hand, suppose $i \in \mathbb{S}_{\text{even}}$ and $j \in \mathbb{S}_{\text{odd}}$ and $i \xrightarrow{n} j$. We know $\alpha \xrightarrow{m} i$ for some even m. Deduce that $\alpha \xrightarrow{m+n} j$, so that $m+n$ is odd, forcing n to be odd. Put another way, we cannot have $i \xrightarrow{n} j$ if n is even; the P^2-chain cannot get from $\mathbb{S}_{\text{even}}$ to $\mathbb{S}_{\text{odd}}$. The argument for $j \xrightarrow{n} i$ is similar.

(v) *Explain why the P^2 chain is aperiodic.*

Invoke Chang notes Lemma 1.38 to show that the set $\{n/2 : n \in \mathbb{N}_\alpha\}$ contains all integers that are large enough. That is, $\alpha \xrightarrow{2n} \alpha$ for all large enough integers n. Under P^2, the state α has period 1. For each j in $\mathbb{S}_{\text{odd}}$ we know there are odd integers k, ℓ for which $\alpha \xrightarrow{k} j$ and $j \xrightarrow{\ell} \alpha$. Thus $j \xrightarrow{k+\ell+2n} j$ for all large enough integers n. It follows that the state j has period 1 under P^2.

(vi) *For each initial distribution ν that concentrates on $\mathbb{S}_{\text{even}}$, show that*

$$\sum\nolimits_{i \in \mathbb{S}_{\text{even}}} |\mathbb{P}_\nu\{X_{2n} = i\} - \lambda_i| \to 0 \qquad \textit{as } n \to \infty.$$

Note that $\lambda P^2 = \mu P = \lambda$. That is, λ is the (unique) stationary probability distribution for the (irreducible, aperiodic) P^2 chain on $\mathbb{S}_{\text{even}}$. The asserted convergence is just the BLT for that chain. [You could argue directly that the chain is positive recurrent, but that is not really needed: the proof of the BLT used positive recurrence just to establish existence of some tationary probability distribution.]

(vii) *For each initial distribution ν that concentrates on $\mathbb{S}_{\text{even}}$, show that*

$$\sum\nolimits_{i \in \mathbb{S}_{\text{odd}}} |\mathbb{P}_\nu\{X_{2n+1} = i\} - \mu_i| \to 0 \qquad \textit{as } n \to \infty.$$

You could argue in essentially the same way as for part (vi), invoking the fact that μ is the (unique) stationary probability distribution for the (irreducible, aperiodic) P^2 chain on $\mathbb{S}_{\text{odd}}$.

Alternatively, you could argue that

$$
\begin{aligned}
&\sum_{i\in\mathcal{S}_{\text{odd}}} |\mathbb{P}_\nu\{X_{2n+1}=i\}-\mu_i| \\
&\qquad = \sum_{i\in\mathcal{S}_{\text{odd}}} |\sum_{j\in\mathcal{S}_{\text{even}}} \mathbb{P}_\nu\{X_{2n}=j\}P(j,i) - \sum_{j\in\mathcal{S}_{\text{even}}} \lambda_j P(j,i)| \\
&\qquad \le \sum_{i\in\mathcal{S}_{\text{odd}}} \textstyle\sum_{j\in\mathcal{S}_{\text{even}}} P(j,i)|\mathbb{P}_\nu\{X_{2n}=j\}-\lambda_j| \\
&\qquad = \sum_{j\in\mathcal{S}_{\text{even}}} |\mathbb{P}_\nu\{X_{2n}=j\}-\lambda_j| \to 0,
\end{aligned}
$$

the last equality following from the fact that $\sum_{i\in\mathcal{S}_{\text{odd}}} P(j,i)=1$ for all $j\in\mathcal{S}_{\text{even}}$.

(viii) *For an arbitrary initial distribution ν on $\mathcal{S}$, describe the behavior of $\mathbb{P}_\nu\{X_n=i\}$ as n tends to infinity. In particular, discuss whether there is a unique stationary probability distribution for the P-chain.*

Write ν as $\gamma\nu_{\text{even}}+(1-\gamma)\nu_{\text{odd}}$, where $\gamma=\nu(\mathcal{S}_{\text{even}})$ and ν_{even} is a probability concentrating on $\mathcal{S}_{\text{even}}$ and ν_{odd} is a probability concentrating on $\mathcal{S}_{\text{odd}}$. Written using the total variation distance (see Chang Definition 1.35), the results from the previous parts of the problem can be summarized as

$$\left\|\nu_{\text{even}}P^{2n}-\lambda\right\|\to 0 \qquad \text{and} \qquad \left\|\nu_{\text{odd}}P^{2n}-\mu\right\|\to 0.$$

The first assertion, together with an argument like the one for the alternative for part (vii), shows that

$$\left\|\nu_{\text{even}}P^{2n}P-\lambda P\right\| \le \left\|\nu_{\text{even}}P^{2n}-\lambda\right\|\to 0$$

That is, $\|\nu_{\text{even}}P^{2n+1}-\mu\|\to 0$. Similarly $\|\nu_{\text{odd}}P^{2n+1}-\lambda\|\to 0$.

Combining these pieces we get

$$
\begin{aligned}
&\left\|\nu P^{2n}-\gamma\lambda-(1-\gamma)\mu\right\|\to 0 \\
&\left\|\nu P^{2n+1}-(1-\gamma\lambda)-\gamma\mu\right\|\to 0
\end{aligned}
$$

If $\gamma=1/2$ it follows that $\|\nu P^n-\pi\|\to 0$ as $n\to\infty$.

If $\widetilde{\pi}$ is another stationary distribution for the P-chain then $\widetilde{\pi}=\widetilde{\pi}P^{2n}=\widetilde{\pi}P^{2n+1}$ for all n, which forces

$$\widetilde{\pi}=\gamma\lambda+(1-\gamma)\mu=(1-\gamma\lambda)+\gamma\mu \qquad \text{for } \gamma=\widetilde{\pi}(\mathcal{S}_{\text{even}}).$$

Thus $\gamma=1/2$ and $\widetilde{\pi}=\frac{1}{2}\lambda+\frac{1}{2}\mu=\pi$. The probability measure π is the unique stationary distribution for the P-chain.