Statistics 251b/551b, spring 2009
Homework #3 solutions

[1] *Section 2.3 of the Chang notes describes a work rule for Andrew: any new arrival is placed on the top of his stack and he then cannot complete any job he was working on. This rule leads to a Markov chain with transition probabilities shown on page 50. Suppose the rule is changed, so that a new arrival is placed at the bottom of the stack and Andrew has probability a of completing any job he is working on, regardless of whether a new paper arrives or not.*

Define A_n to be 1 if a new paper arrives at time n, zero otherwise. Define D_n to be 1 if a job is completed at time n, zero otherwise. (For example, if $X_{n-1} = 3$ and $A_n = D_n = 1$ then $X_n = X_{n-1} + 1 - 1 = 3$, but if $X_{n-1} = 0$ then D_n must be zero, and $X_n = X_{n-1} + A_n$.)

The new transition probabilities are $P(0,1) = p = 1 - P(0,0)$, the same as for Section 2.3 of the Notes, and, for $i \geq 1$,

$$P(i,j) = \begin{cases} \beta = p(1-a) & \text{for } j = i+1 \\ \gamma = (1-p)(1-a) + pa & \text{for } j = i \\ q = (1-p)a & \text{for } j = i-1 \end{cases}$$

Even under the change of rules, the queue size is still a birth and death Markov chain. We can find a stationary probability distribution, if it exists, by solving the 'local balance equations': $\pi_i P(i, i+1) = \pi_{i+1} P(i+1, i)$ for all $i \geq 0$. More precisely,

$$\pi_0 p = \pi_1 q \qquad \text{and} \qquad \pi_i \beta = \pi_{i+1} q \qquad \text{for } i \geq 1.$$

The solution to these equations satisfies $\pi_{i+1} = (p/q)\lambda^{i-1}\pi_0$ for all $i \geq 0$, where

$$\lambda = \frac{\beta}{q} = \frac{p(1-a)}{(1-p)a}$$

Notice that $\lambda < 1$ if $p < a$. In that case, π_0 is well defined by the condition

$$1 = \pi_0 \left(1 + \frac{p}{q} \sum\nolimits_{i \geq 1} \lambda^{i-1}\right) = \pi_0 \left(1 + \frac{p}{q(1-\lambda)}\right) = \pi_0 \left(1 + \frac{p}{a-p}\right)$$

That is, $\pi_0 = 1 - p/a$.

The next three parts of the question asked you to find $\mathbb{P}_\pi\{D_1 = 0\}$ and $\mathbb{P}_\pi\{X_1 = 0, D_1 = 0\}$, with the aim of deducing that X_1 and D_1 are not independent under the $\mathbb{P}_\pi$ distribution. For the first of the probabilities, argue that $D_1 = 1$ if and only if $X_0 \geq 1$ and an independent event with probability a occurs. That is,

$$\mathbb{P}_\pi\{D_1 = 1\} = \mathbb{P}_\pi\{X_0 \geq 1, D_1 = 1\} = (1 - \pi_0)a = p.$$

Thus $\mathbb{P}_\pi\{D_1 = 0\} = 1 - p$. Similarly, $X_1 = 0$ and $D_1 = 0$ if and only if $X_0 = 0$ and $A_1 = 0$, which implies

$$\mathbb{P}_\pi\{X_1 = 0,\ D_1 = 0\} = \mathbb{P}_\pi\{X_0 = 0, A_1 = 0\} = \pi_0(1-p)$$

Unfortunately for DP, the last expression equals $\mathbb{P}_\pi\{X_1 = 0\}\mathbb{P}_\pi\{D_1 = 0\}$; the events $\{X_1 = 0\}$ and $\{D_1 = 0\}$ are independent.

In fact the random variable X_1 and D_1 are independent. To establish this fact it is enough to show that $\mathbb{P}_\pi\{X_1 = k, D_1 = 0\}$ factorizes into $\mathbb{P}_\pi\{X_1 = k\}\mathbb{P}_\pi\{D_1 = 0\}$ for each nonnegative integer k.

The case $k = 0$ has already been covered. For the case $k = 1$ argue that the event $\{X_1 = 1, D_1 = 0\}$ is the disjoint union of the two events $\{X_0 = 0, A_1 = 1\}$ and $\{X_0 = 1, A_1 = 0, D_1 = 0\}$. Thus

$$\begin{aligned}
&\mathbb{P}_\pi\{X_1 = 1, D_1 = 0\} \\
&\quad = \mathbb{P}_\pi\{X_0 = 0, A_1 = 1\} + \mathbb{P}_\pi\{X_0 = 1, A_1 = 0, D_1 = 0\} \\
&\quad = \pi_0 p + \pi_1(1-p)(1-a) \\
&\quad = \pi_1\left(q + (1-p)(1-a)\right) \\
&\quad = \pi_1(1-p) = \mathbb{P}_\pi\{X_1 = 1\}\mathbb{P}_\pi\{D_1 = 0\}.
\end{aligned}$$

For $k \geq 2$ argue that

$$\begin{aligned}
&\mathbb{P}_\pi\{X_1 = k, D_1 = 0\} \\
&\quad = \mathbb{P}_\pi\{X_0 = k, A_1 = 0, D_1 = 0\} + \mathbb{P}_\pi\{X_0 = k-1, A_1 = 1, D_1 = 0\} \\
&\quad = \pi_k(1-p)(1-a) + \pi_{k-1}p(1-a) \\
&\quad = \pi_k\left((1-p) + p/\lambda\right)(1-a) \\
&\quad = \pi_k(1-p) = \mathbb{P}_\pi\{X_1 = k\}\mathbb{P}_\pi\{D_1 = 0\}.
\end{aligned}$$

I also ground out some other cases involving D_1, D_2, and X_2, but kept getting independence. Very strange. I need to think about the reversibility argument some more.