Statistics 251b/551b, spring 2009
Homework #5 solutions

[1] *[20+20 points] Chang Problem 4.6.*

Most of you had no trouble with this question.

[2] *[20+20 points] Chang Problem 4.8.*

It is cleanest if we define $N_n = 2+n(c+d)$ and let X_n denote the number of white balls in urn at nth step. Then

$$\begin{aligned}\mathbb{E}(X_{n+1} \mid X_n = x) &= x + cm_n + d(1-m_n) \qquad \text{where } m_n = x/N_n \\ &= m_n\,(N_n + c - d) + d = m_n N_{n+1} + d(1-2m_n)\end{aligned}$$

so that $\mathbb{E}(M_{n+1} \mid X_n = x) = m_n + (d/N_{n+1})(1-2m_n)$.

If $d = 0$ then we get a martigale. Otherwise we have a “self-reinforcing”process, which I talked about in class (2 March).

[3] *[20 points] Complete the stopping time argument for the gambler’s ruin problem with $p \neq q$, as begun in class on Wednesday 18 February. You may assume that the game end in a finite time, with probability one. That is, you may assume that $\mathbb{P}\{\tau < \infty\} = 1$. Be sure to explain carefully how you pass from the the equality involving $\tau \wedge k$ to the limit.*

Let $W = \{\text{gambler wins}\} = \{\tau < \infty, X_\tau = N\}$. Then

$$\begin{aligned}s^a = \mathbb{E}Z_0 = \mathbb{E}Z_{\tau\wedge k} &\qquad \text{optional stopping} \\ = \mathbb{E}\left(Z_{\tau\wedge k} \mid \tau \le k, X_\tau = 0\right)\mathbb{P}\{\tau \le k, X_\tau = 0\} \\ + \mathbb{E}\left(Z_{\tau\wedge k} \mid \tau \le k, X_\tau = N\right)\mathbb{P}\{\tau \le k, X_\tau = N\} \\ + \mathbb{E}\left(Z_{\tau\wedge k} \mid \tau > k\right)\mathbb{P}\{\tau > k\}\end{aligned}$$

The last sum simplifies to

$$s^0\mathbb{P}\left(\{\tau \le k\} \cap W^c\right) + s^N\mathbb{P}\left(\{\tau \le k\} \cap W\right) + (*)$$

where $(*)$ equals

something bounded $\times$ a probability that tends to zero.

Let k go to ∞, using the fact that $\mathbb{P}\left(\{\tau \le k\} \cap W^c\right)$ converges to $\mathbb{P}W^c$ and $\mathbb{P}\left(\{\tau \le k\} \cap W\right)$ converges to $\mathbb{P}W$. (No formal justification of these two limiting assertions was needed, but some of you noted that they follow by Dominated Convergence.) In the limit we have $s^a = \mathbb{P}W^c + s^N\mathbb{P}W$, which can be solved for $\mathbb{P}W$.