[1] Suppose $Z$ has a standard normal distribution and $x$ is a positive constant.

(i) [5 points] Explain why $1\{Z > x\} \leq \exp (\lambda Z - \lambda x)$ for each $\lambda > 0$.

When $Z \leq x$ the indicator function is zero. The inequality is then trivially true because $\exp(\ldots)$ is always $\geq 0$. When $Z > x$ the indicator function is one and the right-hand side is $\geq 1$ because $\lambda (Z - x) > 0$ and $\exp(\text{nonneg}) \geq 1$.

(ii) [5 points] Deduce that $P\{Z > x\} \leq \exp(\lambda^2/2 - \lambda x)$ for each $\lambda > 0$.

Take expectations of both sides of the inequality from (i). Then note that $P\{Z > x\} = E1\{Z > x\}$ and $Ee^{\lambda Z} = \exp(\lambda^2/2)$, the moment generating function of a $N(0, 1)$.

(iii) [5 points] Deduce that $P\{Z > x\} \leq \exp(-x^2/2)$.

Choose $\lambda = x$ in (ii) to minimize the quadratic in the exponent.


Remember that $\text{cov}(W_a, W_b) = \min(a, b)$. Thus

$$\text{cov}(X_s, X_t) = e^{-s}e^{-t}\text{cov}(W(e^{2s}), W(e^{2t})) = e^{-s-t} \min(e^{2s}, e^{2t})$$

If $s \leq t$ the last expression equals $e^{-s-t+2s}$; if $s > t$ it equals $e^{-s-t+2t}$. In both cases the covariance can be rewritten as $e^{-|s-t|}$.

The increment $\Delta_h = W(e^{2(t+h)}) - W(e^{2t})$ has a $N(0, e^{2(t+h)} - e^{2t})$ distribution independently of any information about what happens up to time $t$. In particular,

$$E(\Delta_h \mid X_t = x) = 0$$

and

$$\text{var}(\Delta_h \mid X_t = x) = e^{2(t+h)} - e^{2t} = e^{2t}(2h + \frac{1}{2}(2h)^2 + \ldots)$$

Thus

$$E(X_{t+h} \mid X_t) = e^{-t-h}E(W(e^{2t}) + \Delta_h \mid X_t)$$

$$= (1 - h + \frac{1}{2}h^2 + \ldots) e^{-t} (W(e^{2t}) + 0)$$

$$\approx (1 - h)X_t \quad \text{if } h \text{ is small.}$$

Equivalently,

$$E(X_{t+h} - X_t \mid X_t = x) = -hx + \text{smaller order terms,}$$
which gives $\mu(x, t) = -x$.

Similarly,

$$\var(X_{t+h} \mid X_t = x) = e^{-2(t+h)} \var(W(e^{2t}) + \Delta_h \mid X_t = x)$$

$$= e^{-2h} e^{-2t} \var(\Delta_h \mid X_t = x)$$

$$= (1 - \ldots) e^{-2t} e^{2t}(2h + \ldots)$$

Note how the $W(e^{2t})$ is treated as a constant, which has no effect on the conditional variance, when we condition on $X_t$. Also, note that any terms contributed by the $e^{-2h}$ beyond the initial constant 1 get swallowed up in the lower order terms, leaving $\sigma^2(x, t) = 2$.

[3] [30 points] Chang Problem 5.13. Hint: Write $\mathbb{P}\{Y(\tau) \leq y\}$ as an integral then differentiate with respect to $y$.

The distribution of $Y(\tau)$ given $\tau = t$ is $N(0, t)$; and (cf. Chang Problem 5.10) the random variable $\tau$ has a distribution with density $f(t) = b(2\pi)^{-1/2} t^{-3/2} \exp(-b^2/2t)$ for $t > 0$. By the conditioning formula when the conditioning variable has a continuous distribution,

$$\mathbb{P}\{Y(\tau) \leq y\} = \int_0^\infty \mathbb{P}\{Y(\tau) \leq y \mid \tau = t\} f(t) \, dt = \int_0^\infty \Phi(y/\sqrt{t}) f(t) \, dt,$$

where $\Phi$ denotes the distribution function for the $N(0, 1)$. Differentiate both sides with respect to $y$ to get the density function $g(y)$ for the distribution of $Y(\tau)$. Note that $d\Phi(y/\sqrt{t})/dy = t^{-1/2} \phi(y/\sqrt{t})$, where $\phi$ is the $N(0, 1)$ density function. That is,

$$g(y) = \int_0^\infty t^{-1/2} \phi(y/\sqrt{t}) f(t) \, dt = \frac{b}{2\pi} \int_0^\infty t^{-2} \exp\left(-\frac{y^2 + b^2}{2t}\right) \, dt.$$  

Temporarily write $z$ for $(y^2 + b^2)/2$. Make the change of variable $s = z/t$ to reduce the last integral to

$$\int_0^\infty z^{-1} \exp(-s) \, ds = z^{-1}.$$  

That is,

$$g(y) = \frac{b}{2\pi} \frac{2}{y^2 + b^2} = \frac{b}{\pi(y^2 + b^2)}.$$  

The distribution of $Y(\tau)/b$ has density $bg(by)$, which is the standard Cauchy density.

[4] [20 points] Let $\{B_t : t \geq 0\}$ be a standard Brownian motion. Find the constant $C$ for which the process $X_t = B_t^3 - CtB_t$ is a martingale.
For a fixed $s > 0$ write $\Delta$ for $B_{t+s} - B_t$. Remember that $\Delta$ has a $N(0, s)$ distribution, independently of anything determined by the information, $\mathcal{F}_t$, determined by what happens up to time $t$. Notice that $\mathbb{E}(\Delta | \mathcal{F}_t) = 0 = \mathbb{E}(\Delta^3 | \mathcal{F}_t)$ by symmetry of the $N(0, s)$ around zero. Thus

$$\mathbb{E} \left( B_{t+s}^3 \mid \mathcal{F}_t \right) = \mathbb{E} \left( (B_t + \Delta)^3 \mid \mathcal{F}_t \right)$$
$$= B_t^3 + 3B_t^2\mathbb{E}(\Delta \mid \mathcal{F}_t) + 3B_t\mathbb{E}(\Delta^2 \mid \mathcal{F}_t) + \mathbb{E}(\Delta^3 \mid \mathcal{F}_t)$$
$$= B_t^3 + 0 + 3B_t s + 0$$

and

$$\mathbb{E}( (t + s)B_{t+s} \mid \mathcal{F}_t ) = (t + s)(B_t + 0).$$

By subtraction,

$$\mathbb{E} \left( B_{t+s}^3 - 3(t + s)B_{t+s} \mid \mathcal{F}_t \right) = B_t^3 - 3tB_t.$$

That is, $M_t = B_t^3 - 3tB_t$ is a martingale.