Statistics 251b/551b, spring 2009
Homework #6 solutions

[1] *Suppose Z has a standard normal distribution and x is a positive constant.*

(i) [5 points] Explain why $\mathbf{1}\{Z > x\} \le \exp(\lambda Z - \lambda x)$ for each $\lambda > 0$.

When $Z \le x$ the indicator function is zero. The inequality is then trivially true because $\exp(\dots)$ is always ≥ 0. When $Z > x$ the indicator function is one and the right-hand side is ≥ 1 because $\lambda(Z - x) > 0$ and $\exp(\text{nonneg}) \ge 1$.

(ii) [5 points] Deduce that $\mathbb{P}\{Z > x\} \le \exp(\lambda^2/2 - \lambda x)$ for each $\lambda > 0$.

Take expectations of both sides of the inequality from (i). Then note that $\mathbb{P}\{Z > x\} = \mathbb{E}\mathbf{1}\{Z > x\}$ and $\mathbb{E}e^{\lambda Z} = \exp(\lambda^2/2)$, the moment generating function of a $N(0, 1)$.

(iii) [5 points] Deduce that $\mathbb{P}\{Z > x\} \le \exp(-x^2/2)$.

Choose $\lambda = x$ in (ii) to minimize the quadratic in the exponent.

[2] *[20+20 points] Chang Problem 5.9.*

Remember that cov $(W_a, W_b) = \min(a, b)$. Thus

$$\text{cov}(X_s, X_t) = e^{-s}e^{-t}\text{cov}\left(W(e^{2s}), W(e^{2t})\right) = e^{-s-t}\min(e^{2s}, e^{2t})$$

If $s \le t$ the last expression equals $e^{-s-t+2s}$; if $s > t$ it equals $e^{-s-t+2t}$. In both cases the covariance can be rewritten as $e^{-|s-t|}$.

The increment $\Delta_h = W(e^{2(t+h)}) - W(e^{2t})$ has a $N(0, e^{2(t+h)} - e^{2t})$ distribution independently of any information about what happens up to time t. In particular,

$$\mathbb{E}\left(\Delta_h \mid X_t = x\right) = 0$$

and

$$\text{var}(\Delta_h \mid X_t = x) = e^{2(t+h)} - e^{2t} = e^{2t}\left(2h + \tfrac{1}{2}(2h)^2 + \dots\right)$$

Thus

$$\begin{aligned}
\mathbb{E}\left(X_{t+h} \mid X_t\right) &= e^{-t-h}\mathbb{E}\left(W(e^{2t}) + \Delta_h \mid X_t\right) \\
&= \left(1 - h + \tfrac{1}{2}h^2 + \dots\right)e^{-t}\left(W(e^{2t}) + 0\right) \\
&\approx (1-h)X_t \qquad \text{if } h \text{ is small.}
\end{aligned}$$

Equivalently,

$$\mathbb{E}\left(X_{t+h} - X_t \mid X_t = x\right) = -hx + \text{ smaller order terms,}$$

which gives $\mu(x,t) = -x$.

Similarly,

$$\begin{aligned}
\text{var}\left(X_{t+h} \mid X_t = x\right) &= e^{-2(t+h)}\text{var}\left(W(e^{2t}) + \Delta_h \mid X_t = x\right) \\
&= e^{-2h}e^{-2t}\text{var}\left(\Delta_h \mid X_t = x\right) \\
&= \left(1 - \dots\right) e^{-2t}e^{2t}(2h + \dots)
\end{aligned}$$

Note how the $W(e^{2t})$ is treated as a constant, which has no effect on the conditonal variance, when we condition on X_t. Also, note that any terms contributed by the e^{-2h} beyond the initial constant 1 get swallowed up in the lower order terms, leaving $\sigma^2(x,t) = 2$.

[3] *[30 points] Chang Problem 5.13. Hint: Write $\mathbb{P}\{Y(\tau) \le y\}$ as an integral then differentiate with respect to y.*

The distribution of $Y(\tau)$ given $\tau = t$ is $N(0,t)$; and (cf. Chang Problem 5.10) the random variable τ has a distribution with density $f(t) = b(2\pi)^{-1/2}t^{-3/2}\exp(-b^2/2t)$ for $t > 0$. By the conditioning formula when the conditioning variable has a continuous distribution,

$$\mathbb{P}\{Y(\tau) \le y\} = \int_0^\infty \mathbb{P}\{Y(\tau) \le y \mid \tau = t\} f(t)\, dt = \int_0^\infty \Phi(y/\sqrt{t}) f(t)\, dt,$$

where Φ denotes the distribution function for the $N(0,1)$. Differentiate both sides with respect to y to get the density function $g(y)$ for the distribution of $Y(\tau)$. Note that $d\Phi(y/\sqrt{t})/dy = t^{-1/2}\phi(y/\sqrt{t})$, where ϕ is the $N(0,1)$ density function. That is,

$$g(y) = \int_0^\infty t^{-1/2}\phi(y/\sqrt{t}) f(t)\, dt = \frac{b}{2\pi}\int_0^\infty t^{-2}\exp\left(-\frac{y^2+b^2}{2t}\right) dt.$$

Temporarily write z for $(y^2+b^2)/2$. Make the change of variable $s = z/t$ to reduce the last integral to

$$\int_0^\infty z^{-1}\exp(-s)\, ds = z^{-1}.$$

That is,

$$g(y) = \frac{b}{2\pi}\frac{2}{y^2+b^2} = \frac{b}{\pi(y^2+b^2)}.$$

The distribution of $Y(\tau)/b$ has density $bg(by)$, which is the standard Cauchy density.

[4] *[20 points] Let $\{B_t : t \ge 0\}$ be a standard Brownian motion. Find the constant C for which the process $X_t = B_t^3 - CtB_t$ is a martingale.*

For a fixed $s > 0$ write Δ for $B_{t+s} - B_t$. Remember that Δ has a $N(0, s)$ distribution, independently of anything determined by the information, $\mathcal{F}_t$, determined by what happens up to time t. Notice that $\mathbb{E}(\Delta \mid \mathcal{F}_t) = 0 = \mathbb{E}(\Delta^3 \mid \mathcal{F}_t)$ by symmetry of the $N(0, s)$ around zero. Thus

$$\begin{aligned}\mathbb{E}\left(B_{t+s}^3 \mid \mathcal{F}_t\right) &= \mathbb{E}\left((B_t + \Delta)^3 \mid \mathcal{F}_t\right) \\ &= B_t^3 + 3B_t^2 \mathbb{E}(\Delta \mid \mathcal{F}_t) + 3B_t \mathbb{E}(\Delta^2 \mid \mathcal{F}_t) + \mathbb{E}(\Delta^3 \mid \mathcal{F}_t) \\ &= B_t^3 + 0 + 3B_t s + 0\end{aligned}$$

and

$$\mathbb{E}((t+s)B_{t+s} \mid \mathcal{F}_t) = (t+s)(B_t + 0).$$

By subtraction,

$$\mathbb{E}\left(B_{t+s}^3 - 3(t+s)B_{t+s} \mid \mathcal{F}_t\right) = B_t^3 - 3tB_t.$$

That is, $M_t = B_t^3 - 3tB_t$ is a martingale.