Statistics 251b/551b, spring 2009
Homework #7 solutions

For the first two questions, A_θ denotes the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ for an angle θ.

[1] *Suppose W is a random vector that is uniformly distributed on the set $\{w \in \mathbb{R}^2 : |w| = 1\}$. Let $\mathbb{E}W = \mu$, a vector with components μ_1 and μ_2, and var$(W) = S$.*

(i) [5 points] Explain why $A_\theta W$ has the same distribution as W, for each fixed θ.

The matrix A_θ rotates vectors in $\mathbb{R}^2$ clockwise through an angle θ. For an arc L of length ℓ on the unit circle,

$$\mathbb{P}\{W \in L\} = \frac{\ell}{2\pi}$$

Let L_θ denote the arc $\{A_{-\theta}w : w \in L\}$. It also has length ℓ and

$$\mathbb{P}\{A_\theta W \in L\} = \mathbb{P}\{W \in L_\theta\} = \frac{\ell}{2\pi},$$

which shows that $A_\theta W$ is also uniformly distributed around the circle.

(ii) [5 points] Explain why $\mathbb{E}(A_\theta W) = A_\theta \mu$ and var$(A_\theta W) = A_\theta$var$(W)A'_\theta$.

Many of you just quoted these as standard results for random vectors. Both can be derived by writing out the vector $A_\theta W$ or the matrix $A_\theta W W' A'_\theta$ and taking expectations of each component. If you have not seen the derivation, start from somewhere like:
`http://en.wikipedia.org/wiki/Covariance_matrix`

As $A_\theta W$ has the same distribution as W, we have $\mu = A_\theta \mu$ and

$$\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} := S = \text{var}(W) = A_\theta S A'_\theta$$

The matrix S is symmetric because $v_{12} = \text{cov}(W_1, W_2) = v_{21}$.

(iii) [10 points] By an appropriate choice of θ, deduce that $\mu = 0$ and $S = \frac{1}{2}I_2$, where I_2 is the 2×2 identity matrix.

With $\theta = \pi/2$ we have $A_{\pi/2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It follows that

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \mu = A_{\pi/2}\mu = \begin{pmatrix} \mu_2 \\ -\mu_1 \end{pmatrix}$$

These equalities imply $\mu_1 = \mu_2 = -\mu_1$, so that $\mu = 0$. Similarly,

$$S = A_{\pi/2} S A'_{\pi/2} = \begin{pmatrix} v_{22} & v_{21} \\ -v_{12} & v_{11} \end{pmatrix}$$

Thus $v_{11} = v_{22}$ and $v_{12} = v_{21} = -v_{12}$, forcing $v_{12} = 0$. The factor $1/2$ comes from $v_{11} + v_{22} = \mathbb{E}(W_1^2 + W_2^2) = 1$.

[2] *[20 points] Let $\{W_t : t \geq 0\}$ be a two-dimensional Brownian motion, that is, its components $\{X_t : t \geq 0\}$ and $\{Y_t : t \geq 0\}$ are independent standard Brownian motions. Explain why $\{A_\theta W_t : t \geq 0\}$, for a fixed θ, is also a two-dimensional Brownian motion.*

The two-dimensional Brownian motion is characterized by the facts that it is Gaussian with continuous sample paths, and $\mathbb{E}W_t = 0$, and

$$\text{cov}(W_s, W_t) = \begin{pmatrix} \mathbb{E}(X_s X_t) & \mathbb{E}(X_s Y_t) \\ \mathbb{E}(X_t Y_s) & \mathbb{E}(Y_s Y_t) \end{pmatrix} = (s \wedge t) I_2.$$

The process $\widetilde{W}_t = A_\theta W_t$ is also Gaussian with continuous sample paths, because both properties are preserved by linear combinations. Also $\mathbb{E}\widetilde{W}_t = A_\theta \mathbb{E} W_t = 0$ and

$$\text{cov}(\widetilde{W}_s, \widetilde{W}_t) = A_\theta \text{cov}(W_s, W_t) A_\theta' = (s \wedge t) A_\theta A_\theta' = (s \wedge t) I_2.$$

The $\widetilde{W}$ process is also a two-dimensional Brownian motion.

[3] *Suppose $\{M_t : 0 \leq t \leq 1\}$ is a martingale with continuous sample paths. That is, M_t is determined by the information, $\mathcal{F}_t$, available at time t and $\mathbb{E}\left(M_t - M_s \mid \mathcal{F}_s\right) = 0$ for each pair of times $0 \leq s < t \leq 1$. Equivalently, $\mathbb{E}\left((M_t - M_s)W\right) = 0$ for each random variable W that depends only on the information $\mathcal{F}_s$.*

Suppose also that there is some process $\{A_t : 0 \leq t \leq 1\}$, with continuous sample paths $A(t, \omega)$ that are increasing in t, such that $N_t = M_t^2 - A_t$ is a martingale. (Necessarily, A_t is determined by the $\mathcal{F}_t$ information.) For simplicity, assume $M_0 = A_0 = 0$.

(i) [10 points] Let G be a grid of time points $0 = t_0 < t_1 < \cdots < t_n \leq 1$. Write $\Delta_i M$ for $M(t_{i+1}, \omega) - M(t_i, \omega)$. Define $\Delta_i N$ and $\Delta_i A$ similarly. Show that

$$\mathbb{E}\left((\Delta_i M)^2 - \Delta_i A \mid \mathcal{F}_{t_i}\right) = 0 \qquad \textit{for } i = 0, 1, \ldots, n-1.$$

Hint: What do you know about $\mathbb{E}\left(\Delta_i N \mid \mathcal{F}_{t_i}\right)$?

Expand the square to get

$$\Delta_i N = (M_{t_i} + \Delta_i M)^2 - (A_{t_i} + \Delta_i A) - M_{t_i}^2 + A_{t_i} = 2 M_{t_i} \Delta_i M + (\Delta_i M)^2 - \Delta_i A$$

The martingale property of N gives $\mathbb{E}(\Delta_i N \mid \mathcal{F}_{t_i}) = 0$. The facts that M is a martingale and M_{t_i} depends only on $\mathcal{F}_{t_i}$ information give

$$\mathbb{E}(M_{t_i} \Delta_i M \mid \mathcal{F}_{t_i}) = M_{t_i} \mathbb{E}(\Delta_i M \mid \mathcal{F}_{t_i}) = 0.$$

(ii) [10 points] Suppose H_G is a simple process, that is,

$$H_G(s,\omega) = \sum\nolimits_{0\le i<n} h_i(\omega)\mathbf{1}\{t_i < s \le t_{i+i}\}$$

where h_i is a random variable that depends only on $\mathcal{F}_{t_i}$ information. (It will behave like a constant when you take expectations conditional on $\mathcal{F}_{t_i}$.) Define

$$Y_G(1) = \int_0^1 H_G(s)dM(s) = \sum\nolimits_{0\le i<n} h_i(\omega)\Delta_i M.$$

Show that $\mathbb{E}Y_G(1) = 0$ and $\mathbb{E}Y_G^2(1) = \mathbb{E}\left(\int_0^1 H_G^2(s,\omega)dA(s)\right)$. [[Remember that $\int_0^1 f(s)dA(s) = \sum_i f_i\Delta_i A$ if $f = \sum_{0\le i<n} f_i\mathbf{1}\{t_i < s \le t_{i+1}\}$.]] Hint: Part (i) should help with terms like $\mathbb{E}h_i^2(\Delta_i M)^2$.

For the expectation: $\mathbb{E}Y_G(1) = \sum_{0\le i<n}\mathbb{E}(h_i(\omega)\Delta_i M) = 0$ because h_i depends only on $\mathcal{F}_{t_i}$ information and $\mathbb{E}(\Delta_i M \mid \mathcal{F}_{t_i}) = 0$. The zero expected value allows us to write the variance of $Y_G(1)$ as

$$\mathbb{E}Y_G(1)^2 = \sum\nolimits_{0\le i<n}\mathbb{E}(h_i^2(\Delta_i M)^2) + 2\sum\nolimits_{0\le i<j\le n}\mathbb{E}(h_i h_j \Delta_i M \Delta_j M)$$

All the cross-product terms vanish because $h_i h_j \Delta_i M$ depends only on $\mathcal{F}_{t_j}$ information, if $i < j$, and $\mathbb{E}(\Delta_j M \mid \mathcal{F}_{t_j}) = 0$. The result from part (i) simplifies the other sum:

$$\sum\nolimits_{0\le i<n}\mathbb{E}(h_i^2(\Delta_i M)^2) = \sum\nolimits_{0\le i<n}\mathbb{E}(h_i^2\Delta_i A)$$

For the final step, note that $H_G^2 = \sum_{0\le i<n} h_i^2\mathbf{1}\{t_i < s \le t_{i+i}\}$ so that $\int_0^1 H_G^2\, dA = \sum_{0\le i<n} h_i^2\Delta_i A$.

(iii) [10 points] For each fixed t in $[0,1]$, show that

$$\begin{aligned} H_G(s,\omega)\mathbf{1}\{0 < s \le t\} &= \sum\nolimits_{0\le i<k} h_i(\omega)\mathbf{1}\{t_i < s \le t_{i+1}\} \\ &\qquad + h_k\mathbf{1}\{t_k < s \le t\} \qquad \textit{if } t_k < t \le t_{k+1} \\ &= \sum\nolimits_{0\le i<n} h_i(\omega)\mathbf{1}\{t_i\wedge t < s \le t_{i+1}\wedge t\}, \end{aligned}$$

a simple process defined for a slightly different grid of points.

The second form follows from the equality of indicator functions,

$$\mathbf{1}\{0 < s \le t\}\mathbf{1}\{t_i < s \le t_{i+1}\} = \mathbf{1}\{t_i\wedge t < s \le t_{i+1}\wedge t\}$$

(The inequalities appearing on the left-hand side are equivalent to the inequalities appearing on the right-hand side.) The first form then follows from

$$\mathbf{1}\{t_i\wedge t < s \le t_{i+1}\wedge t\} = \begin{cases} \mathbf{1}\{t_i < s \le t_{i+1}\} & \text{if } t > t_{i+1} \\ \mathbf{1}\{t_i < s \le t\} & \text{if } t_i < t \le t_{i+1} \\ 0 & \text{if } t \le t_i \end{cases}$$

(iv) [10 points] Define

$$Y_G(t) = \int_0^t H_G(s)\mathbf{1}\{0 < s \le t\}\, dM(s) \qquad \textit{for } 0 \le t \le 1.$$

Show that $\{Y_G(t) : 0 \le t \le 1\}$ has continuous sample paths. Hint: Use the second form in part (iii) for the integrand.

From (iii),

$$Y_G(t) = \sum\nolimits_{0 \le i < n} h_i(\omega)\,(M(t_{i+1} \wedge t) - M(t_i \wedge t))$$

Each $t \wedge t_i$ is a continuous function of t and $M(s)$ is a continuous function of s. A finite sum of continuous function is also continuous.

(v) [10 points] Show that $\{Y_G(t) : 0 \le t \le 1\}$ is a martingale. Hint: If $s < t$, you may suppose $s = t_j$ and $t = t_k$ for some $j < k$. (If s and t were not grid points, you could refine the grid by adding them in. None of the preceding definitions would be significantly changed.)

If $s = t_j$ and $t = t_k$ then

$$Y_G(s) = \sum\nolimits_{0 \le i < j} h_i \Delta_i M$$
$$Y_G(t) = \sum\nolimits_{0 \le i < k} h_i \Delta_i M$$

If W depends only on $\mathcal{F}_{t_j}$ information then

$$\mathbb{E}\,(Y_G(t) - Y_G(s))\,W = \sum\nolimits_{j \le i < k} \mathbb{E}\,(h_i W \Delta_i M) = 0$$

because $h_i W$ depends only on $\mathcal{F}_{t_i}$ information and $\mathbb{E}(\Delta_i \mid \mathcal{F}_{t_i}) = 0$.