Statistics 251b/551b, spring 2009
Homework #8 solutions

For this sheet you might find it useful to know the Cauchy-Schwarz inequality: if X and Y are random variables then $(\mathbb{E}XY)^2 \le (\mathbb{E}X^2)(\mathbb{E}Y^2)$. A similar inequality holds for integrals of real-valued functions:

<1>
$$\left|\int_0^1 f(s)g(s)\,ds\right|^2 \le \int_0^1 f(s)^2 ds \int_0^1 g(s)^2 ds.$$

As a special case, when $Y \equiv 1$ or $g(s) \equiv 1$, we have

<2>
$$(\mathbb{E}X)^2 \le \mathbb{E}X^2 \qquad \textit{and} \qquad \left(\int_0^1 f(s)\,ds\right)^2 \le \int_0^1 f(s)^2\,ds.$$

When combined, the two Cauchy-Schwarz inequalities give a bound for stochastic processes. Suppose $F(s,\omega)$ and $G(s,\omega)$ are stochastic processes indexed by s in $[0,1]$. Then

$$\begin{aligned}
\left(\int_0^1 \mathbb{E}F_sG_s\,ds\right)^2 &\le \left(\int_0^1 |\mathbb{E}F_sG_s|\,ds\right)^2 \\
&\le \left(\int_0^1 \sqrt{\mathbb{E}F_s^2}\sqrt{\mathbb{E}G_s^2}\,ds\right)^2 \\
&\le \int_0^1 \mathbb{E}F_s^2\,ds \int_0^1 \mathbb{E}G_s^2\,ds.
\end{aligned}$$
<3>

See the note at the end of the sheet for the proof of the Cauchy-Schwarz inequality.

[1] Suppose $\{B_t : 0 \le t \le 1\}$ is a standard Brownian motion and

$$H_n(s,\omega) = \sum\nolimits_i h_i(\omega)\mathbf{1}\{t_i < s \le t_{i+1}\} \qquad \textit{for } n = 1,2,\dots$$

is a sequence of simple predictable processes for which

<4>
$$\int_0^1 \mathbb{E}|H_n(s) - H(s)|^2\,ds \to 0 \qquad \textit{as } n \to \infty,$$

where H is a process for which $\int_0^1 \mathbb{E}H_s^2\,ds < \infty$. Here $G_n : 0 = t_0 < t_1 < \cdots < t_n = 1$ is a sequence of grids for which $\max_i \delta_i \to 0$, where $\delta_i = t_{i+1} - t_i$. As usual, write $\Delta_i B$ for $B_{t_{i+1}} - B_{t_i}$. Show that $\sum_i h_i(\Delta_i B)^2$ converges to $\int_0^1 H_s\,ds$, in an appropriate sense as n tends to infinity, by the following steps.

(i) [20 points] Define $\xi_i = (\Delta_i B)^2 - \delta_i$. Show that

$$\sum\nolimits_i h_i(\Delta_i B)^2 = \int_0^1 H_n(s)\,ds + Z_n$$

where $Z_n = \sum_i h_i \xi_i$.

With H_n as defined, $\int_0^1 H_n(s)\,ds = \sum_i h_i \delta_i$. The rest is subtraction.

(ii) [20 points] Show that

$$\mathbb{E}\left|\int_0^1 H_n(s)\,ds - \int_0^1 H(s)\,ds\right|^2 \le \int_0^1 \mathbb{E}|H_n(s) - H(s)|^2\,ds \to 0.$$

Inequality <2> for integrals gives

$$\left(\int_0^1 H_n(s) - H(s)\,ds\right)^2 \le \int_0^1 \left(H_n(s) - H(s)\right)^2\,ds$$

Take expectations, then move the $\mathbb{E}$ inside the integral. Assumption <4> completes the argument.

(iii) [20 points] Explain why

$$\int_0^1 \mathbb{E}H_n(s)^2\,ds \le 4\int_0^1 \mathbb{E}H(s)^2\,ds + 4\int_0^1 \mathbb{E}|H_n(s) - H(s)|^2\,ds,$$

so that $\int_0^1 \mathbb{E}H_n(s)^2\,ds$ stays bounded as n tends to infinity.

Use the inequality $(a+b)^2 \le 4a^2 + 4b^2$ [actually the 4's could be replaced by 2's] with $a = H(s)$ and $b = H_n(s) - H(s)$, then take $\int_0^1 \mathbb{E}$ of both sides. Assumption <4> and the finiteness of $\int_0^1 \mathbb{E}H_s^2\,ds < \infty$ complete the argument.

(iv) [20 points] Show that $\mathbb{E}Z_n = 0$ and

$$\mathbb{E}Z_n^2 = \sum\nolimits_i \mathbb{E}\left(h_i^2\xi_i^2\right) \le C\max\nolimits_i \delta_i \int_0^1 \mathbb{E}H_n(s)^2\,ds$$

for some constant C. Deduce that $\mathbb{E}Z_n^2 \to 0$ as $n \to \infty$.

By the properties of Brownian motion, $\mathbb{E}(\xi_j \mid \mathcal{F}_{t_j}) = 0$, which implies $\mathbb{E}(\xi_j W_j) = 0$ for each W_j that depends only on $\mathcal{F}_{t_j}$ information. [Alternatively, use the facts that ξ_j is independent of every random variable that depends only on $\mathcal{F}_{t_j}$ information and $\mathbb{E}\xi_j = 0$.]

Expand the sum for Z_n then take expectations to get

$$\mathbb{E}Z_n^2 = \sum\nolimits_i \mathbb{E}\left(h_i^2\xi_i^2\right) + 2\sum\nolimits_{i<j} \mathbb{E}\left(h_i\xi_i h_j\xi_j\right).$$

The cross-product term for $i < j$ vanishes because $h_i \xi_i h_j$ depends only on $\mathcal{F}_{t_j}$ information.

For the terms with $i = j$, independence of ξ_i^2 and h_i^2 gives

$$\mathbb{E}\left(h_i^2 \xi_i^2\right) = \mathbb{E}\left(h_i^2\right) \mathbb{E}\left(\xi_i^2\right).$$

Use the fact that Δ_i has the same distribution as $\sqrt{\delta_i}\eta$, with $\eta \sim N(0,1)$, to see that ξ^2 has the same distribution as $\delta_i^2(\eta^2 - 1)^2$. Let $C = \mathbb{E}(\eta^2 - 1)^2$, which is finite. Then $\mathbb{E}\xi_i^2 = \delta_i^2 C$ and

$$\mathbb{E}Z_n^2 = \sum\nolimits_i \mathbb{E}\left(h_i^2\right) C\delta_i^2 \leq (\max_i \delta_i) C \mathbb{E} \int_0^1 \sum\nolimits_i h_i^2 \mathbf{1}\{t_i < s \leq t_{i+1}\}\, ds$$

The conclusion follows from (iii) and the assumption that $\max_i \delta \to 0$.

[2] *[10+20 points] Let $Y = \int_0^1 B_s\, ds$. Show that $\mathbb{E}Y = 0$ and find var(Y). Hint: For the second part, write Y^2 as $\int_0^1 \int_0^1 B_s B_t\, ds\, dt$.*

Both results come from taking expectations inside integral signs.

$$\mathbb{E}Y = \int_0^1 \mathbb{E}B_s\, ds = 0 \qquad \text{because } \mathbb{E}B_s = 0.$$

Similarly,

$$\text{var}(Y) = \mathbb{E}(Y^2) = \int_0^1 \int_0^1 \mathbb{E}(B_s B_t)\, ds\, dt.$$

Substitute $\min(s,t)$ for $\mathbb{E}(B_s B_t)$. Then argue by symmetry that the last expression equals

$$2 \int_0^1 \int_0^t s\, ds\, dt = \int_0^1 t^2\, dt = 1/3.$$

[3] *Suppose $\{B_t : 0 \leq t \leq 1\}$ is a standard Brownian motion.*

(i) [20 points] Use the Itô formula to find $\int_0^1 B_s^3\, dB_s$.

For $f(x,y) = x^4$ we have $f_x(x,y) = 4x^3$ and $f_{xx}(x,y) = 12x^2$ and $f_y(x,y) = 0$. Itô gives

$$B_t^4 - 0^4 = \int_0^t 4B_s^3\, dB_s + \int_0^t 6B_s^2\, ds$$

Thus

$$\int_0^t B_s^3\, dB_s = \frac{1}{4} B_t^4 - \frac{3}{2} \int_0^t B_s^2\, ds.$$

(ii) [20 points] Grind out the result from part (i) by first principles. Start from the expansion of $B_{t_{i+1}}^4 - B_{t_i}^4$. Kill off terms that need to be killed by calculating

expected values and variances. You may invoke the result from Problem 1 when needed.

Start from a grid $0 = t_0 < t_1 < \cdots < t_n = 1$ with $t_i = i/n$. To simplify notation, write Z_i for B_{t_i} and Δ_i for $\Delta_i B = Z_{i+1} - Z_i$ and δ_i for $t_{i+1} - t_i = 1/n$. Remember that $\Delta_i \sim N(0, \delta_i)$ and it is independent of $Z_1, \ldots, Z_i$.

We need to find a telescoping sum. Note that

$$Z_{i+1}^4 - Z_i^4 = (Z_i + \Delta_i)^4 - Z_i^4 = 4Z_i^3\Delta_i + 6Z_i^2\Delta_i^2 + 4Z_i\Delta_i^3 + \Delta_i^4$$

Sum.

$$4\sum_{i<n} Z_i^3\Delta_i = B_1^4 - 6\sum_{i<n} Z_i^2\delta_i - 6U_n - 4V_n - W_n$$

$$\text{where } U_n = \sum_{i<n} Z_i^2(\Delta_i^2 - \delta_i)\,,\ V_n = \sum_{i<n} Z_i\Delta_i^3\,, \quad W_n = \sum_{i<n} \Delta_i^4$$

The term $\sum_{i<n} Z_i^2\delta_i$ converges to $\int_0^1 B_s^2\,ds$ as n tends to infinity. The other three terms should go to zero, in some sense, leaving an expression that agrees with the Itô formula.

For U_n, first note that $\xi_i := \Delta_i^2 - \delta_i$ is distributed like $\delta_i(\eta^2 - 1)$, where $\eta \sim N(0,1)$; it has zero expected value and $\text{var}(\xi_i) = \mathbb{E}\xi_i^2 = C\delta_i^2$ where $C = \mathbb{E}(\eta^2 - 1)^2$. By independence of ξ_i and Z_i,

$$\mathbb{E}U_n = \sum\nolimits_{i<n} \mathbb{E}Z_i^2\mathbb{E}\xi_i = 0.$$

Similarly, ξ_j and $Z_i^2\xi_i Z_j^2$ are independent if $i < j$ and Z_i^2 is distributed like $\sqrt{t_i}\eta$, which leads to

$$\begin{aligned}
\mathbb{E}U_n^2 &= \sum\nolimits_{i<n} \mathbb{E}\left(Z_i^4\right)\mathbb{E}\left(\xi_i^2\right) + 2\sum\nolimits_{i<j<n} \mathbb{E}(Z_i^2\xi_i Z_j^2)\mathbb{E}(\xi_j) \\
&= \sum\nolimits_{i<n} t_i^2\mathbb{E}\left(\eta^4\right)C\delta_i^2 + 0 \\
&\le E(\eta^4)C/n \to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

Thus the contribution from U_n goes away in the limit.

For V_n a calculation with expected values suffices.

$$\begin{aligned}
\mathbb{E}|V_n| &\le \sum\nolimits_{i<n} \mathbb{E}|Z_i|\mathbb{E}|\Delta_i|^3 \\
&\le \sum\nolimits_{i<n} \sqrt{t_i}\mathbb{E}|\eta|\delta_i^{3/2}\mathbb{E}|\eta|^3 \\
&= O(n^{-1/2}) \to 0.
\end{aligned}$$

Similarly, since $W_n \ge 0$,

$$\mathbb{E}|W_n| = \sum\nolimits_{i<n} \mathbb{E}\Delta_i^4 = \sum\nolimits_{i<n} \delta_i^2\mathbb{E}\eta^4 = O(n^{-1}) \to 0.$$

Proof of the Cauchy-Schwarz inequality.

Write c for $\sqrt{\mathbb{E}X^2}$ and d for $\sqrt{\mathbb{E}Y^2}$. Expand the left-hand side of the inequality

$$\mathbb{E}\left|\frac{X}{c} \pm \frac{Y}{d}\right|^2 \geq 0$$

into $\mathbb{E}X^2/c^2 + \mathbb{E}Y^2/d^2 \pm 2\mathbb{E}(XY)/(cd)$ then rearrange to get

$$2 \pm 2\mathbb{E}(XY)/(cd) \geq 0,$$

which implies $\pm E(XY) \leq cd$. The argument for the other form of the inequality is similar.