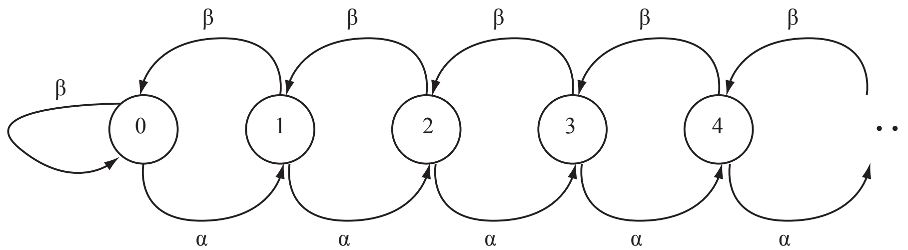
In class on Wednesday 23 January I discussed a Markov chain with state space $\mathcal{S} = \{0, 1, 2, \dots\}$ and transition probabilities

$$P(i,j) = \begin{cases} \alpha & \text{if } j = i+1 \\ \beta & \text{if } i \geq 1 \text{ and } j = i-1 \\ \beta & \text{if } i = j = 0 \end{cases}$$

where $\alpha + \beta = 1$ and $0 < \alpha < 1$.



I showed that the chain is recurrent if $\beta > 1/2$ and transient if $\beta < 1/2$. For $\beta = 1/2$ I referred you to Example 1.27 of the Chang notes for the analysis of an analogous random walk on the integers.

This handout summarizes the argument for the $\beta \neq 1/2$ cases.

Recurrence when $\beta > 1/2$

Intuitively, the reason for recurrence is: the chain is pushed towards 0 from every state. To formalize the idea it is enough to construct a stationary probability distribution π. The equations for stationarity are

<1> $$\pi_0 = \beta\pi_0 + \beta\pi_1$$

<2> $$\pi_j = \alpha\pi_{j-1} + \beta\pi_{j+1} \qquad \text{for } j \geq 1.$$

Equation <1> implies

$$\pi_1 = \gamma\pi_0 \qquad \text{where } \gamma = \alpha/\beta.$$

Equation <2> implies

$$\pi_{j+1} = \beta^{-1}\pi_j - \gamma\pi_{j-1}.$$

Repeated substitution, starting with

$$\pi_2 = \beta^{-1}\pi_1 - \gamma\pi_0 = \left(\beta^{-1} - 1\right)\gamma\pi_0 = \gamma^2\pi_0,$$

or a formal induction, shows that the only possible solution is $\pi_j = \gamma^j\pi_0$. To ensure that $\sum_{i\in\mathcal{S}}\pi_0 = 1$ we must have $\pi_0\sum_{i\geq 0}\gamma^i = 1$. This would be impossible to achieve if γ were ≥ 1. Fortunately the assumption $\beta > 1/2$ implies $\gamma < 1$ so that $\pi_0 = 1 - \gamma$ and $\pi_i = \gamma^i(1-\gamma)$ is the stationary probability distribution.

Transience when $\beta < 1/2$

For this case, intuition suggests that the chain is being pushed out towards infinity. For large n, the random variable X_n should be large with high probability, for any starting state.

To formalize the idea I will use what you will later recognize as a martingale method. See Example 2.6 in the Stat 241/541 notes for an analogous analysis of the gambler's ruin problem.

For a fixed s in $(0,1)$, which will be specified soon, calculate $\mathbb{E}_0 s^{X_n}$ in two different ways. First use conditioning on the value of X_{n-1}:

$$\begin{aligned}
&\mathbb{E}_0 s^{X_n}\\
&= \sum\nolimits_{j\ge 0} \mathbb{P}_0\{X_{n-1}=j\}\mathbb{E}_0\left(s^{X_n} \mid X_{n-1}=j\right)\\
&= \mathbb{P}_0\{X_{n-1}=0\}\left(\beta s^0+\alpha s^1\right) + \sum\nolimits_{j\ge 1}\mathbb{P}_0\{X_{n-1}=j\}\left(\beta s^{j-1}+\alpha s^{j+1}\right)\\
&\le \left(\beta s^{-1}+\alpha s\right)\sum\nolimits_{j\ge 0}\mathbb{P}_0\{X_{n-1}=j\}s^j\\
&\qquad\qquad \text{because } \beta+\alpha s < \beta s^{-1}+\alpha s \text{ if } 0<s<1\\
&= (\beta s^{-1}+\alpha s)\mathbb{E}_0 s^{X_{n-1}}.
\end{aligned}$$

The value $s = \sqrt{\beta/\alpha}$ minimizes $\beta s^{-1}+\alpha s$ with a minimum of $\rho := 2\sqrt{\alpha\beta}$, which is <1 because $\beta \ne 1/2$. With that choice for s we have

$$\mathbb{E}_0 s^{X_n} \le \rho\mathbb{E}_0 s^{X_{n-1}} \le \rho^2\mathbb{E}_0 s^{X_{n-2}} \le \dots \le \rho^n\mathbb{E}_0 s^{X_0} = \rho^n.$$

For the second method, just condition on X_n itself to get the usual expression for the expected value,

$$\mathbb{E}_0 s^{X_n} = \sum\nolimits_{j\ge 0}\mathbb{P}_0\{X_n=j\}s^j \ge \mathbb{P}_0\{X_n=0\}.$$

Thus $\mathbb{P}_0\{X_n=0\} \le \rho^n$, which tends to zero as $n\to\infty$.

Finally, show that $\mathbb{E}_0 N_0 < \infty$ to establish transience.

$$\begin{aligned}
\mathbb{E}_0 N_0 &= \mathbb{E}_0\sum\nolimits_{n\in\mathbb{N}}\mathbb{I}\{X_n=0\}\\
&= \sum\nolimits_{n\in\mathbb{N}}\mathbb{P}_0\{X_n=0\}\\
&\le \sum\nolimits_{n\in\mathbb{N}}\rho^n\\
&< \infty \qquad \text{because } 0<\rho<1.
\end{aligned}$$