Optimal stopping

Consider a finite set of random variables $\{Z_t : t \in T\}$ where $T = \{1, 2, \ldots, N\}$, which you observe sequentially. You need to choose one of Z_t's—call it the σth—to receive a payoff. Imagine that, at each time $t < N$, you have two choices:

(i) Accept Z_t based on what you have seen so far, namely the values of $Z_{1,t} := \{Z_1, \ldots, Z_t\}$. In this case σ equals t. You then cannot change your mind, no matter what you see after time t.

(ii) Reject Z_t (irrevocably) then go on to the next $(t+1)$ step.

Of course, if you reject all Z_t for $1 \le t < N$ then you are stuck with Z_N and $\sigma = N$. Your reward will be $\mathbb{E}Z_\sigma$.

Remark. Note that σ is a stopping time because $\mathbb{I}\{\sigma = t\}$ is a function of $Z_{1,t}$.

Problem: Find a stopping time σ, taking values in T, that maximize $\mathbb{E}Z_\sigma$.

Case 1: Z itself is a supermartingale

If $Z_1, Z_2, \ldots$ is a supermartingale then $\mathbb{E}Z_1 \ge \mathbb{E}Z_\sigma$ for all T-valued stopping times σ. You should always accept Z_1.

Case 2: Z is bounded above by a supermartingale Y that hits Z

Suppose there exists a supermartingale $Y_1, Y_2, \ldots$ with $Y_t \ge Z_t$ for each t. Suppose also that there exists a stopping time τ (with values in T) for which $Z_\tau = Y_\tau$ (with probability one). Then there is no point in stopping later than τ:

$$\begin{aligned}
&\mathbb{E}\left(Z_{\tau\wedge\sigma} - Z_\sigma\right)\\
&\quad = \mathbb{E}\left((Z_\tau - Z_\sigma)\,\mathbb{I}\{\tau \le \sigma\}\right) + \mathbb{E}\left((Z_\sigma - Z_\sigma)\,\mathbb{I}\{\tau > \sigma\}\right)\\
&\quad \ge \mathbb{E}\left((Y_\tau - Y_\sigma)\,\mathbb{I}\{\tau \le \sigma\}\right) + 0 \qquad \text{because } Z_\tau = Y_\tau \text{ and } Z_\sigma \le Y_\sigma\\
&\quad = \mathbb{E}\left(Y_{\tau\wedge\sigma} - Y_\sigma\right)\\
&\quad \ge 0 \qquad \text{by supermartingale property.}
\end{aligned}$$

Case 3: Be guided by the smallest supermartingale dominating Z

Of all the supermartingales W for which $W_t \ge Z_t$ for $t \in T$ there is a smallest. It is defined by $Y_N := Z_N$ and (inductively)

<1> $$Y_t := \max\left(Z_t, \mathbb{E}(Y_{t+1} \mid Z_{1,t})\right) \qquad \text{for } t = N-1, N-2, \ldots, 1$$

Remark. This procedure is sometimes called backward induction. The Y process itself is often called the Snell envelope of the Z process (Snell, 1952).

You should check that

(i) Y is a supermartingale: each random variable Y_t depends only on $Z_{1,t}$ and $Y_t \geq \mathbb{E}(Y_{t+1} \mid Z_{1,t})$ (with probability one)

(ii) if $W_1, W_2, \dots, W_N$ is a supermartingale with $W_t \geq Z_t$ for all t then $W_t \geq Y_t$ (with probability one): start from $W_N \geq Z_N = Y_N$ then work backwards

The optimal stopping time τ is then defined by

<2> $$\tau := \min\{t : Z_t = Y_t\}$$

Case 2 ensures that $\mathbb{E}Z_{\sigma\wedge\tau} \geq \mathbb{E}Z_\sigma$ for all stopping times σ taking values in T. It remains only to show that $\mathbb{E}Z_\tau \geq \mathbb{E}Z_{\sigma\wedge\tau}$ for each stopping time σ.

<3> **Lemma.** *With Y as defined in <1> and τ as in <2>, the process*

$$M_t := Y_{t\wedge\tau} \qquad \text{for } t \in T$$

is a martingale.

Proof

$$\begin{aligned}
&\mathbb{E}\left(M_{t+1} - M_t \mid Z_{1,t}\right)\\
&\quad = \mathbb{E}\left((Y_\tau - Y_\tau)\mathbb{I}\{\tau \leq t\} + (Y_{t+1} - Y_t)\mathbb{I}\{\tau > t\} \mid Z_{1,t}\right)\\
&\quad = 0 + \mathbb{I}\{\tau > t\}\left(\mathbb{E}(Y_{t+1} \mid Z_{1,t}) - Y_t\right)\\
&\qquad\qquad \text{as } \mathbb{I}\{\tau > t\} \text{ and } Y_t \text{ are functions of } Z_{1,t}
\end{aligned}$$

Then use the fact that $Y_t > Z_t$ if $\tau > t$ so that

$$Y_t = \max\left(Z_t, \mathbb{E}(Y_{t+1} \mid Z_{1,t})\right) = \mathbb{E}(Y_{t+1} \mid Z_{1,t}) \qquad \text{if } \tau > t.$$

□

Finish Case 3:

$$\begin{aligned}
\mathbb{E}Z_\tau = \mathbb{E}Y_\tau &= \mathbb{E}M_\tau && \text{by definition of } \tau \text{ and } M\\
&= \mathbb{E}M_\sigma && \text{by optional stopping for martingales}\\
&= \mathbb{E}Y_{\tau\wedge\sigma}\\
&\geq \mathbb{E}Z_{\tau\wedge\sigma}.
\end{aligned}$$

Applications

In class I will talk about the simple example where $Z_1, \dots, Z_N$ are independent random variables, each distributed Uniform(0,1). I will also introduce the Secretary (a.k.a. Princess) problem, as descibed by Ferguson (1989, page 242):

1. There is one secretarial position available.
2. The number [N] of applicants is known.
3. The applicants are interviewed sequentially in random order, each order being equally likely.
4. It is assumed that you can rank all the applicants from best to worst without ties. The decision to accept or reject an applicant must be based only on the relative ranks of those applicants interviewed so far.
5. An applicant once rejected cannot later be recalled.
6. You are very particular and will be satisfied with nothing but the very best. (That is, your payoff is 1 if you choose the best of the [N] applicants and 0 otherwise.)

Notes

For more about optimal stopping and games see Ferguson (2008). For a Markov chain approach to the "Princess problem" (also known as the "Secretary problem") see Billingsley (1986, pages 110, 130–137).

A clear exposition of the Princess/Secretary problem, including the connections between the supermartingale and Markov chain approaches, would make a moderately challenging project for this course. Of course you would have to explain how "First, one shows that attention can be restricted to the class of rules that for some integer $r > 1$ rejects the first $r - 1$ applicants" (Ferguson, 1989, page 282).

References

Billingsley, P. (1986). *Probability and Measure* (Second ed.). New York: Wiley. (third edition, 1995).

Ferguson, T. S. (1989). Who solved the secretary problem? *Statistical Science 4*(3), 282–289. (with discussion, pages 289–296).

Ferguson, T. S. (2008). Optimal stopping and applications. (Unpublished text, available at his web site: http://www.math.ucla.edu/~tom/).

Snell, J. L. (1952). Applications of martingale system theorems. *Transactions of the American Mathematical Society 73*(2), 293–312.