Existence of stationary distributions

Suppose a Markov chain with state space $\mathcal{S}$ is irreducible and recurrent.

Let i be an arbitrarily chosen but fixed state. For each $j \in \mathcal{S}$ define

$$\begin{aligned}\lambda_j &:= \mathbb{E}_i\,(\text{number of visits to } j \text{ during a cycle around } i)\\ &= \mathbb{E}_i \sum\nolimits_{n\in\mathbb{N}} \mathbb{I}\{X_n = j,\, T_i \ge n\}\\ &= \sum\nolimits_{n\in\mathbb{N}} \mathbb{P}_i\{X_n = j,\, T_i \ge n\}\end{aligned}$$

where, as usual, T_i is the first time (after time 0) that the chain visits state i. Note that $\lambda_i = 1$ because the cycle ends with the first return, at time T_i, to state i.

First I'll show that

<1> $$\lambda_j = \sum\nolimits_{k\in\mathcal{S}} \lambda_k P(k,j) \qquad \text{for each } j \in \mathcal{S}$$

Then I'll show that $\sum_{j\in\mathcal{S}} \lambda_j = \mathbb{E}_i T_i$. If the state i is positive recurrent then $\pi_j = \lambda_j/\mathbb{E}_i T_i$ defines a stationary probability distribution for the chain.

> **Remark.** By construction, $\pi_i = 1/\mathbb{E}_i T_i$. As you know from HW2.2, if i is positive recurrent then every state in $\mathcal{S}$ must also be positive recurrent. I could repeat the construction with any other state i' taking over the role of i to get another stationary probability distribution $\{\pi'_j : j \in \mathcal{S}\}$ for which $\pi'_{i'} = 1/\mathbb{E}_{i'} T_{i'}$.
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> It might appear that the chain has many different stationary distributions. However, the Basic Limit Theorem will force the stationary distribution to be unique. That is, we must have $\pi_i = \pi'_{i'} = 1/\mathbb{E}_{i'} T_{i'}$. The unique stationary distribution for an irreducible, positive recurrent chain is given by $\pi_j = 1/\mathbb{E}_j T_j$ for every $j \in \mathcal{S}$.

The key idea behind <1> is that the event $\{T_i \ge n\}$ only depends on information about the chain before time n. More precisely, T_i is always ≥ 1 and

$$\{T_i \ge n\} = \{X_1 \ne i, X_2 \ne i, \dots, X_{n-1} \ne i\} \qquad \text{for } n \ge 2.$$

Thus

$$\begin{aligned}\sum\nolimits_{k\in\mathcal{S}} \lambda_k P(k,j) &= \sum\nolimits_{k\in\mathcal{S}} \sum\nolimits_{n\ge1} \Big(\mathbb{P}_i\{X_n = k,\, T_i \ge n\} \times\\ &\qquad\qquad \mathbb{P}_i\{X_{n+1} = j \mid X_n = k,\, T_i \ge n\}\Big)\\ &= \sum\nolimits_{k\in\mathcal{S}} \sum\nolimits_{n\ge1} \mathbb{P}_i\{X_n = k,\, T_i \ge n, X_{n+1} = j\}\end{aligned}$$

The event $\{T_i \geq n, X_{n+1} = j\}$ equals $\cup_{k\in\mathcal{S}}\{X_n = k,\, T_i \geq n, X_{n+1} = j\}$ and for different k the events $\{X_n = k,\, T_i \geq n, X_{n+1} = j\}$ are disjoint. The last sum simplifies to

$$\sum\nolimits_{n\geq 1} \mathbb{P}_i\{T_i \geq n, X_{n+1} = j\} = \sum\nolimits_{n\geq 1} \mathbb{P}_i\{T_i \geq n+1, X_{n+1} = j\} + \sum\nolimits_{n\geq 1} \mathbb{P}_i\{T_i = n, X_{n+1} = j\}.$$

On the right-hand side the first sum can be rewitten as

$$\sum\nolimits_{m\geq 2} \mathbb{P}_i\{T_i \geq m, X_m = j\},$$

which is just λ_j minus $\mathbb{P}_i\{T_i \geq 1, X_1 = j\} = P(i,j)$. The other sum equals

$$\sum\nolimits_{n\geq 1} \mathbb{P}_i\{X_{n+1} = j \mid T_i = n\}\mathbb{P}_i\{T_i = n\} = P(i,j)\sum\nolimits_{n\geq 1} \mathbb{P}_i\{T_i = n\},$$

which adds back the missing $P(i,j)$. (The sum of the $\mathbb{P}_i\{T_i = n\}$ terms equals $\mathbb{P}_i\{T_i < \infty\} = 1$.)

So much for <1>. The rest is easy. The random variable $\sum_{n\geq 1} \mathbb{I}\{T_i \geq n\}$ counts one for each n for which $T_i \geq n$. In other words, $\sum_{n\geq 1} \mathbb{I}\{T_i \geq n\} = T_i$. It follows that

$$\begin{aligned}\sum\nolimits_{j\in\mathcal{S}} \lambda_j &= \mathbb{E}_i \sum\nolimits_{n\in\mathbb{N}} \sum\nolimits_{j\in\mathcal{S}} \mathbb{I}\{X_n = j,\, T_i \geq n\} \\ &= \mathbb{E}_i \sum\nolimits_{n\in\mathbb{N}} \mathbb{I}\{T_i \geq n\} \\ &\qquad\qquad \text{because } \textstyle\sum_{j\in\mathcal{S}} \mathbb{I}\{X_n = j\} = 1 \text{ for every } n \\ &= \mathbb{E}_i T_i.\end{aligned}$$