You asked:

We say a process $X_0, X_1, \dots$ satisfies the Markov property if

$$\mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n,\, X_{n-1} = i_{n-1},\, \dots, X_0 = i_0)$$
$$= \mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n). \tag{1}$$

So for example,

$$\mathbb{P}(X_3 = i_3 \mid X_2 = i_2, X_1 = i_1, X_0 = i_0) = \mathbb{P}(X_3 = i_3 \mid X_2 = i_2).$$

But what about $\mathbb{P}(X_3 = i_3 \mid X_2 = i_2, X_0 = i_0)$? Using the law of total probability, I get

$$\begin{aligned}
&\mathbb{P}(X_3 = i_3 \mid X_2 = i_2, X_0 = i_0)\\
&\quad= \sum_{i\in\mathcal{S}} \mathbb{P}(X_3 = i_3 \mid X_2 = i_2, X_1 = i, X_0 = i_0) * \mathbb{P}(X_1 = i \mid X_2 = i_2, X_0 = i_0)\\
&\quad= \sum_{i\in\mathcal{S}} \mathbb{P}(X_3 = i_3 \mid X_2 = i_2) * \mathbb{P}(X_1 = i \mid X_2 = i_2, X_0 = i_0)\\
&\quad= \mathbb{P}(X_3 = i_3 \mid X_2 = i_2) * \sum_{i\in\mathcal{S}} \mathbb{P}(X_1 = i \mid X_2 = i_2, X_0 = i_0)\\
&\quad= \mathbb{P}(X_3 = i_3 \mid X_2 = i_2) * 1\\
&\quad= \mathbb{P}(X_3 = i_3 \mid X_2 = i_2).
\end{aligned}$$

So generalizing the argument, may one assume that

$$\mathbb{P}(X_{n+1} = i_{n+1} \mid X_n = i_n, info) = \mathbb{P}(X_{n+1} = i_{n=1} \mid X_n = i_n) \tag{2}$$

where info stands for possible conditions on the values of X_j for $j < n$?

I think JC used definition (1) because it looks friendlier than (2), which is how I defined the Markov property. As you observed, (2) is more useful and it follows from (1). Indeed, if $Y = (X_0, \dots, X_{n-1})$ and $\mathcal{Y}$ is any subset of $\mathcal{S}^n$ then

$$\begin{aligned}
&\mathbb{P}\{X_{n+1} = i_{n+1} \mid X_n = i_n,\, Y \in \mathcal{Y}\}\\
&= \sum_{y\in\mathcal{Y}} \mathbb{P}\{X_{n+1} = i_{n+1} \mid X_n = i_n,\, Y = y,\, Y \in \mathcal{Y}\} \times \mathbb{P}\{Y = y \mid X_n = i_n,\, Y \in \mathcal{Y}\}
\end{aligned}$$

On the right-hand side, the first factor equals

$$\mathbb{P}\{X_{n+1} = i_{n+1} \mid X_n = i_n,\, Y = y\} = \mathbb{P}\{X_{n+1} = i_{n+1} \mid X_n = i_n\}$$

by (1). That factor comes outside the sum, leaving a sum that is equal to $\mathbb{P}\{Y \in \mathcal{Y} \mid X_n = i_n,\, Y \in \mathcal{Y}\} = 1$, as in your argument.

dp 20 jan 2013