Statistics 251/551 spring 2013
Homework # 2
Due: Wednesday 30 January

If you are not able to solve a part of a problem, you can still get credit for later parts: Just assume the truth of what you were unable to prove in the earlier part.

[1] Consider an irreducible Markov chain with a finite state space $\mathcal{S}$. Show that every state i is positive recurrent (that is, $\mathbb{E}_i T_i < \infty$) by following these steps.

(i) (5 points) Explain why there is a positive integer N and a $\delta > 0$ for which $\mathbb{P}_j\{T_i \le N\} \ge \delta$ for all $j \in \mathcal{S}$. Hint: For each $j \ne i$ there is a path from j to i that is taken with strictly positive $\mathbb{P}_j$ probability. What if $j = i$?

(ii) (5 points) For each $k \in \mathbb{N}$ write $\mathcal{U}_k$ for the set of times $\{n \in \mathbb{N} : (k-1)N < n \le kN\}$. Let W denote the first k for which there is a visit to i during $\mathcal{U}_k$. Explain why $T_i \le NW$.

(iii) (5 points) Intuitively, from part (i), within each $\mathcal{U}_k$ time block the chain should visit state i with probability at least δ, no matter what has happened up to time $(k-1)N$. To formalize this intuition, define

$$V_k = \{\text{chain visits } i \text{ during block } \mathcal{U}_k\} = \cup_{n \in \mathcal{U}_k}\{X_n = i\}.$$

Show that $\mathbb{P}_i(V_\ell^c \mid V_1^c \cap V_2^c \cap \cdots \cap V_{\ell-1}^c) \le 1 - \delta$ for each $\ell \ge 2$. (Hint: Condition on $X_{(\ell-1)N}$.) Deduce that $\mathbb{P}_i\{W \ge k\} \le (1-\delta)^{k-1}$ for each k in $\mathbb{N}$.

(iv) (5 points) If the last inequality were actually an equality for each k then W would have a geometric distribution, which has a finite expected value. In some sense, W is smaller than a geometric, so W should also have a finite expected value. Prove that $\mathbb{E}_i W < \infty$. Hint: First explain why $W = \sum_{k \in \mathbb{N}} \mathbb{I}\{W \ge k\}$.

[2] Suppose $i \rightsquigarrow j$ (that is, state j is accessible from state i). Suppose also that $\tau := \mathbb{E}_i T_i < \infty$. Show that $\mathbb{E}_j T_j < \infty$ by the following steps.

Remember that $\theta = \mathbb{P}_i\{X_1 = i_1, X_2 = i_2, \dots, X_k = j\} > 0$ for some sequence of states $i_1, i_2, \dots, i_k = j$. Write $T_i^{(1)}, T_i^{(2)}, \dots$ for the lengths of successive cycles that start from i. That is, if starting from state i, the first return to i occurs at time $T_i^{(1)}$, the second at time $T_i^{(1)} + T_i^{(2)}$, and so on. If starting from state j, the successive visits to i occur at times T_i, $T_i + T_i^{(1)}$, $T_i + T_i^{(1)} + T_i^{(2)}, \dots$. Also, write F_m for the event that the mth cycle (starting from i) begins with visits to states $i_1, i_2, \dots, i_k$ in that order.

(i) (5 points) Explain why $\mathbb{E}_i T_i^{(m)} \ge \theta \mathbb{E}_i(T_i^{(m)} \mid F_m)$ and $\mathbb{E}_i(T_i^{(m)} \mid F_m) = k + \mathbb{E}_j T_i$. Deduce that $\mathbb{E}_j T_i < \infty$.

(ii) (5 points) If the chain starts in state i, explain why

$$T_j \le T_i^{(1)} + \sum_{m \ge 2} \left(T_i^{(m)} \mathbb{I}(F_1^c \cap F_2^c \dots F_{m-1}^c) \right).$$

Hint: If $m = 3$ is the first cycle for which F_m occurs, why is T_j less than $T_i^{(1)} + T_i^{(2)} + T_i^{(3)}$?

(iii) (5 points) By taking $\mathbb{E}_i$ expectations of both sides of the inequality from the previous part, and by using independence between what happens in each cycle, deduce that

$$\mathbb{E}_i T_j \le \tau \left(1 + (1-\theta) + (1-\theta)^2 + \dots \right) < \infty$$

(iv) (5 points) Write $T_i + S_j$ for the first time after T_i that the chain visits state j. Show that

$$\mathbb{E}_j T_j \le \mathbb{E}_j(T_i + S_j) \le \mathbb{E}_j T_i + \mathbb{E}_i T_j < \infty.$$