Statistics 251/551 spring 2013
Homework # 3
Due: Wednesday 30 January

If you are not able to solve a part of a problem, you can still get credit for later parts: Just assume the truth of what you were unable to prove in the earlier part.

[1] (10 points) Find the transition probability matrix Q^* for the bivariate chain $Z_n^* = (X_n Y_n^*)$ that appeared in the coupling proof of the BLT. *It would be a good idea to check that*

$$\sum\nolimits_{z_1 \in \mathcal{S}\times\mathcal{S}} Q^*(z_0, z_1) = 1 \qquad \text{for each } z_0.$$

[2] (15 points) Chang Exercise 1.17.

[3] Consider once again the one-sided random walk described on Wednesday 23 January (and on the onesideRW.pdf handout). Suppose $\beta > 1/2$. You know that the chain has a stationary probability distribution. The chain is recurrent. In fact (see bonus question) it is positive recurrent. Find the value $\tau := \mathbb{E}_1 T_0$ by these steps.

(i) (10 points) Explain why τ must be finite if the chain is positive recurrent.

(ii) (10 points) Show that $\mathbb{E}_k T_0 = k\tau$ for each $k \geq 1$.

(iii) (10 points) Set up an equation for τ by conditioning on the first step, then solve. *Your solution should tend to infinity as β decreases to $1/2$.*

[4] (bonus points) You know that an irreducible Markov chain that has a stationary probability distribution $\pi = \{\pi_i : i \in \mathcal{S}\}$ must be recurrent. Use the BLT to show that such a chain must actually be positively recurrent. Here are some ideas that might help.

For some state i suppose $\mathbb{E}_i T_i = +\infty$.

(i) Let $V_N = \sum_{n \leq N} \mathbb{I}\{X_n = i\}$ denote the number of visits to state i in the first N steps. Use the BLT to show that $\mathbb{E}_i V_N / N \to \pi_i$ as $N \to \infty$.

(ii) For each positive integer M, show that $V_N \geq M$ iff $\sum_{n \leq M} T_i^{(n)} \leq N$, where the $T_i^{(n)}$'s are the successive cycle times.

(iii) With $\mathbb{P}_i$ probability one, $\sum_{n \leq M} T_i^{(n)}/M \to \infty$. (Why?) Deduce, for each $\epsilon > 0$, that

$$\mathbb{P}_i\{V_N/N > \epsilon\} \to 0 \qquad \text{as } N \to \infty.$$

(iv) Deduce that $\mathbb{E}_i(V_N/N) < 2\epsilon$ for N large enough.

(v) What does that tell you about π_i?

(vi) In fact, is it possible to have $\pi_j = 0$ for at least one j in $\mathcal{S}$?