Statistics 251/551 spring 2013
Homework # 5
Due: Wednesday 27 February

If you are not able to solve a part of a problem, you can still get credit for later parts: Just assume the truth of what you were unable to prove in the earlier part.

[1] Suppose σ and τ are stopping times for a "flow of information" $W_0, W_1, \dots$. Which of the following are necessarily stopping times? In each case give a proof or counterexample.

(i) (5 points) $\max(\sigma, \tau)$

(ii) (5 points) $\min(\sigma, \tau)$

(iii) (5 points) $\tau + \sigma$

(iv) (5 points) $(\tau - \sigma)^+ := \max(\tau - \sigma, 0)$

[2] Consider a random walk on the integers $\mathbb{Z}$ defined by $W_n = W_0 + \sum_{1 \le i \le n} \xi_i$ where $W_0, \xi_1, \xi_2, \dots$ are independent random variables with

$$\mathbb{P}\{\xi_i = +1\} = 1/2 = \mathbb{P}\{\xi_i = -1\}.$$

For each m in $\mathbb{Z}$ define

$$\sigma_m = \begin{cases} \min\{n \in \mathbb{N} : W_n = m\} & \\ +\infty & \text{if } W_n \neq m \text{ for all } n \in \mathbb{N} \end{cases}$$

and $\tau := \min(\sigma_0, \sigma_N)$ where $N := a + b$, with a and b positive integers. As in the class presentation of the gambler's ruin problem, write V for the event $\{\tau = \sigma_N\}$, which corresponds to A winning. Write θ_i for $\mathbb{P}(V \mid W_0 = i)$ and λ_i for $\mathbb{E}(\tau \mid W_0 = i)$ if $0 \le i \le N$.

(i) (10 points) Show that $W_0, W_1, \dots$ is a martingale. Use the optional sampling theorem for martingales to show that $\theta_a = a/N$. (Don't forget to start with $\tau \wedge k$ for a positive integer k.)

(ii) (10 points) Define $Z_n := W_n^2 - n$. Show that $Z_0, Z_1, \dots$ is a martingale.

(iii) (10 points) Use the optional sampling theorem for martingales to show that $a^2 = N^2\theta_a - \lambda_a$ then deduce that $\lambda_a = ab$. Hint: $Z_{\tau\wedge k} = W_{\tau\wedge k}^2 - \tau \wedge k$.

(iv) (10 points) Write δ_i for $\lambda_i - \lambda_{i-1}$. Use Markov chain methods to show that $\delta_{i+1} = \delta_i - 2$ for $i = 1, 2, \dots, N-1$.

(v) (10 points) Show that $\lambda_i = \sum_{j=1}^{i} \delta_j = i\delta_1 - (i-1)i$ for $1 \le i \le N$. Deduce that $\lambda_i = i(N - i)$ for $0 \le i \le N$, in agreement with (iii).