Statistics 251/551 spring 2013
Homework # 6
Due: Wednesday 6 March

If you are not able to solve a part of a problem, you can still get credit for later parts: Just assume the truth of what you were unable to prove in the earlier part.

[1] (15 points) Suppose

(a) $S_0, S_1, \ldots$ is a supermartingale for a flow of information $W_0, W_1, \ldots$

(b) σ and τ are stopping times for which $0 \leq \sigma \leq \tau \leq N$, where N is a fixed (finite) positive integer

(c) B is a nonnegative random variable for which $B\mathbb{I}\{\sigma = j\}$ depends only on $W_{0,j}$ for $0 \leq j \leq N$. That is,

$$B = \sum\nolimits_{0 \leq j \leq N} b_j(W_{0,j})\mathbb{I}\{\sigma = j\}$$

for some nonnegative b_j functions.

Show that $\mathbb{E}S_\sigma B \geq \mathbb{E}S_\tau B$. Explain each step in your reasoning. Hint: Write $\mathbb{E}(S_\tau - S_\sigma)B$ as a double sum over t (for the increments) and j (for the value of σ).

[2] (10 points) Near the top of page 2 of the optimal.pdf handout I asserted that the Y process defined at the bottom of page 1 is actually the smallest supermartingale for which $Y_t \geq Z_t$ for each t. Prove that fact.

[3] In class I discussed the simple example where $Z_1, \ldots, Z_N$ are independent random variables, each distributed Uniform(0,1), and we seek a stopping time σ for which $\mathbb{E}Z_\sigma$ is maximized. Write M_N for $\max_{i \leq N} Z_i$.

(i) (10 points) Express $\mathbb{E}M_N$ as a function of N.

(ii) (10 points) For $N = 5$ calculate the constants C_i for which $Y_i = \max(Z_i, C_i)$ and

$$\tau := \min\{i \leq N : Z_i = Y_i\}$$

defines the optimal stopping time. (Please explain your reasoning and display your results in a form that is easy for us to read.)

(iii) (10 points) Calculate $\mathbb{E}Z_\tau$. Again, explain your reasoning; a single number will not suffice.

(iv) (5 points) Explain why your answer from (i) for $N = 5$ is bigger than your answer from (ii).