Statistics 251/551 spring 2013
Homework # 7
Due: Wednesday 3 April

If you are not able to solve a part of a problem, you can still get credit for later parts: Just assume the truth of what you were unable to prove in the earlier part.

[1] Suppose $B = \{B_t : t \geq 0\}$ is a standard Brownian motion. In class I showed, for each fixed $\theta \in \mathbb{R}$, that the process

$$M_t = \exp\big(\theta B_t - \theta^2 t/2)\big) \qquad \text{for } t \geq 0$$

is a martingale for the flow of information $\text{info}_t = B_{0,t}$.

(i) (10 points) Expand M_t as a power series in θ for each fixed t. Find the coefficient of θ^k for $k = 0, 1, 2, 3$.

(ii) (20 points) Write Z_t for the coefficient of θ^3 from part (i). Show by direct calculation (as in class) that Z is a martingale.

[2] (bonus points) Suppose $X = \{X_t : t \geq 0\}$ and $Y = \{Y_t : t \geq 0\}$ are independent standard Brownian motions. Define a 2-dimensional Brownian motion by $W_t = (X_t, Y_t)$.

(i) Explain why $W_0 = 0$ and why W has continuous sample paths and independent increments. Find the (bivariate) distribution of $W_t - W_s$ for fixed $t > s$.

(ii) For a fixed real θ define orthogonal unit vectors $e_1 = (\cos\theta, \sin\theta)$ and $e_2 = (-\sin\theta, \cos\theta)$. Express W in the new coordinate system as

$$W_t = X_t^* e_1 + Y_t^* e_2 \qquad \text{for } t \geq 0.$$

Show that X^* and Y^* are also independent standard Brownian motions.

(iii) For a fixed $r > 0$ define $\tau = \inf\{t : \|W_t\| = r\}$. Explain why W_τ is uniformly distributed on $\{w \in \mathbb{R}^2 : \|w\| = r\}$.