Statistics 251/551 spring 2013
Homework # 8
Due: Wednesday 17 April

This homework will step you through some of the basic ideas of stochastic calculus. The homework problems themselves are boxed so that you don't miss them.

1. Nonrandom case

Start with a simple, nonrandom case. Suppose f and g are continuous functions defined at least on some interval $[a, b]$. What should $\int_a^b f dg$ mean? Typically integrals are defined as limits of finite sums, each defined by a finite grid of points

$$\mathbb{G}: \quad a = t_0 < t_1 < \cdots < t_k = b.$$

Define the $\mathbb{G}$-increments of the function g as $\Delta_i g := g(t_i) - g(t_{i-1})$ for $i = 1, \ldots, k$. Define the corresponding approximating sum as

<1> $$\mathbb{S}_{\mathbb{G}}(f, dg) = \sum_{i=1}^{k} f(t_{i-1})\Delta_i g$$

Remark. I have cunningly evaluated the f function at the left endpoint of the interval that defines $\Delta_i g$. For nonrandom functions the cunningness is wasted. For stochastic processes, it will make a difference.

It is not so hard to understand why the the approximating sums converge to a finite limit if the function g has ***bounded variation*** on the interval $[a, b]$, that is, if there exists a finite constant $C := V(g, [a, b])$ for which $\sum_i |\Delta_i g| \le C$ for every grid, no matter how finely spaced the grid points are.

Remark. If g is nondecreasing then $\sum_i |\Delta_i g| = \sum_i \Delta_i g = g(b) - g(a)$ for every grid. Thus g has bounded variation on $[a, b]$. A similar argument works if g can be written as a difference of two nondecreasing functions. In fact, g has bounded variation if and only if it can be written as a difference of two nondecreasing functions.

Suppose that g has bounded variation. Let $\mathbb{G}'$ be a finer grid obtained by subdividing each $\mathbb{G}$-interval:

$$t_{i-1} = t_{i-1,0} < t_{i-1,1} < \ldots t_{i-1,m} = t_i \qquad \text{for each } i.$$

Write $\Delta_{i-1,j} g$ for $g(t_{i-1,j}) - g(t_{i-1,j-1})$. Then

$$\mathbb{S}_{\mathbb{G}'}(f, dg) = \sum_i \sum_j f(t_{i-1,j-1})\Delta_{i-1,j} g$$

so that

<2>
$$
\begin{aligned}
|\mathcal{S}_{\mathbb{G}}(f,dg) - \mathcal{S}_{\mathbb{G}'}(f,dg)| &= |\sum\nolimits_i \sum\nolimits_j (f(t_{i-1}) - f(t_{i-1,j-1}))\,\Delta_{i-1,j}g| \\
&\le \sum\nolimits_i \sum\nolimits_j |f(t_{i-1}) - f(t_{i-1,j-1})|\;|\Delta_{i-1,j}g|
\end{aligned}
$$

If $\mathbb{G}$ is fine enough that f varies by at most ϵ over each $[t_{i-1}, t_i]$ interval then the last sum is bounded above by $\epsilon V(g, [a, b])$. The bound can be made arbtrarily small by making $\mathbb{G}$ fine enough. If you know about Cauchy sequences of real numbers you should be able to turn the bound into a rigorous argument that the approximating sums converge to some finite limit as the grids get finer.

2. Difficulties with Brownian motion

If X and Y are stochastic processes with continuous smple paths, and if each sample path of Y has bounded variation, then the integral $\int_a^b X\,dY$ can be defined by applying the preceeding construction pathwise. Unfortunately, Brownian motion sample paths don't have bounded variation. For simplicity, take $[a, b]$ as the unit interval $[0, 1]$.

[1]

Suppose $B = \{B_t : 0 \le t \le 1\}$ is a standard Brownian motion. For a fixed k let $t_i = i/k$ for $i = 0, \ldots, k$. Write $\Delta_i B$ for $B(t_i) - B(t_{i-1})$. Define $V_k = \sum_{i=1}^k |\Delta_i B|$.

(i) (10 points) Show that $\mathbb{E}V_k \to \infty$ as $k \to \infty$ but

$$
\text{var}\,(V_k) = \sum\nolimits_i \text{var}(|\Delta_i B|)
$$

stays bounded by 1. Hint: $\text{var}(|X|) \le \mathbb{E}X^2$.

(ii) (5 points) Deduce that $\mathbb{P}\{V(B, [0, 1]) = \infty\} = 1$. Hint: Use Tchebychev for $\mathbb{P}\{V_k \ge M\}$ for each constant M.

Fortunately, the martingale properties of Brownian motion offer another approach to defining $\int_0^1 X\,dB$, for various stochastic processes X. The key to the construction is finiteness of the ***quadratic variation***. In the notation of the previous problem, if $Q_k = \sum_i \Delta_i^2 B \quad (= \sum_i (\Delta_i B)^2)$ then $\mathbb{E}Q_k = 1$ and

$$
\text{var}(Q_k) = \sum\nolimits_i \text{var}(\Delta_i^2 B) \le \sum\nolimits_i \mathbb{E}\Delta_i^4 B \to 0 \qquad \text{as } k \to \infty
$$

because each $\Delta_i B$ is $N(0, k^{-1})$ distributed. Compare with Chang page 195.

3. Stochastic integral with respect to Brownian motion

Suppose $X = \{X_t : 0 \le t \le 1\}$ is another stochastic process, with continuous sample paths, for which X_t is a function of $B_{0,t}$.

Let $\mathbb{G}$ and $\mathbb{G}'$ be grids as in Section 1 (but now $a = 0$ and $b = 1$). Define $\mathcal{S}_{\mathbb{G}}(X, dB)$ as in <1>, with f replaced by X and g replaced by B. Define $\mathcal{S}_{\mathbb{G}'}(X, dB)$ similarly. Then, as in <2>,

$$|\mathcal{S}_{\mathbb{G}}(X, dB) - \mathcal{S}_{\mathbb{G}'}(X, dB)| = |\sum_i \sum_j (X(t_{i-1}) - X(t_{i-1,j-1}))\, \Delta_{i-1,j} B|$$

Instead of taking the absolute value inside the sum, take the expected value of squared sum.

[2]

(10 points) Show that

<3>
$$\begin{aligned}\mathbb{E}|\mathcal{S}_{\mathbb{G}}(X, dB) - \mathcal{S}_{\mathbb{G}'}(X, dB)|^2 &= \sum_{i,j} \mathbb{E}\,(X(t_{i-1}) - X(t_{i-1,j-1}))^2\, \Delta^2_{i-1,j} B \\ &= \sum_{i,j} \mathbb{E}\,(X(t_{i-1}) - X(t_{i-1,j-1}))^2\, \delta_{i-1,j}\end{aligned}$$

where $\delta_{i-1,j} = \mathbb{E}\Delta^2_{i-1,j} B = t_{i-1,j} - t_{i-1,j-1}$. Hint: $\Delta_{i-1,j} B$ is independent of $B_{0,s}$ for $s = t_{i-1,j-1}$.

Intuitively, if the $\mathbb{G}$-spacing is fine enough then each $|X(t_{i-1}) - X(t_{i-1,j-1})|$ should be small with high probability, by sample path continuity. If we could turn that property into a bound,

$$\max_{i,j} \mathbb{E}\,(X(t_{i-1}) - X(t_{i-1,j-1}))^2 < \epsilon,$$

then the final sum in <3> would be bounded by ϵ. We could then argue that, as the grid gets finer, $\mathcal{S}_{\mathbb{G}}(X, dB)$ gets arbitrarily close in some mean-squared error sense to a random variable that then defines $\int_0^1 X\, db$. If you know about Cauchy sequences in spaces of square-integrable functions you should be able to make this argument rigorous.

More precisely, the quantity in <3> can be written as

$$\int_0^1 \mathbb{E}\,(X_{\mathbb{G}}(t) - X_{\mathbb{G}'}(t))^2\, dt$$

where

$$\begin{aligned}X_{\mathbb{G}}(t) &= \sum_i X(t_{i-1}) \mathbb{I}\{t_{i-1} \le t < t_i\} \\ X_{\mathbb{G}'}(t) &= \sum_{i,j} X(t_{i-1,j-1}) \mathbb{I}\{t_{i-1,j-1} \le t < t_{i,j}\}.\end{aligned}$$

If the assumptions on X are enough to guarantee that

$$\int_0^1 \mathbb{E}\,(X_{\mathbb{G}}(t) - X(t))^2\, dt \to 0 \qquad \text{as the grid gets finer}$$

then

$$\int_0^1 \mathbb{E}\left(X_{\mathbb{G}}(t) - X_{\mathbb{G}'}(t)\right)^2 dt$$
$$\leq 2\int_0^1 \mathbb{E}\left(X_{\mathbb{G}}(t) - X(t)\right)^2 dt + 2\int_0^1 \mathbb{E}\left(X_{\mathbb{G}'}(t) - X(t)\right)^2 dt$$
$$\to 0,$$

which would also lead to the existence of the limit (in the sense of mean-squared error) of the approximations $\mathcal{S}_{\mathbb{G}}(X, dB)$.

4. Contributions from quadratic variation

In class I ended up with expressions like

$$A_{\mathbb{G}} = \sum\nolimits_i Z(t_{i-1})\Delta_i^2 B$$

for grids $\mathbb{G}$ as in Section 1 and processes Z. These sums also converge to a random limit under mild assumptions on the Z process. Once again assume Z_t is a function of $B_{0,t}$ and the sample paths are continuous.

For simplicity assume $t_i = i/k$ for each i. Simplify notation by abbreviating B_{0,t_i} to $\mathcal{F}_i$. Remember that $\mathbb{E}\left(\Delta_i^2 B \mid \mathcal{F}_{i-1}\right) = \delta_i := t_i - t_{i-1}$. Define $\xi_i := \Delta_i^2 B - \delta_i$. The following problem gives a simple condition under which $\sum_i Z(t_{i-1})\Delta_i^2 B$ converges to $\int_0^1 Z_t\, dt$ in some probabilistic sense.

[3]

Suppose $|Z_t| \leq C$ for all t (and every sample path), where C is a finite constant.

(i) (10 points) Explain why $\mathbb{E}\left(\sum_i Z(t_{i-1})\xi_i\right)^2 \to 0$ as $k \to \infty$.

(ii) (5 points) Explain why $\sum_i Z(t_{i-1})\delta_i \to \int_0^1 Z_t\, dt$ pathwise.

Remark. The boundedness assumption on Z can be removed by defining stopping times $\tau_m = \inf\{t : |Z_t - Z_0| \geq m\}$ for each positive integer m. One then replaces Z_t by $Z_t - Z_0$ and each t_i by $t_i \wedge \tau_m$ and argues as in the problem to get convergence in some sense to $Z_0 + \int_0^1 Z_{t\wedge\tau_m}\, dt$, for each fixed m. Finally one lets m tend to infinity.

Similar tricks with stopping times can be used to eliminate other boundedness assumptions, at the cost of weakening the convergence to an "in probability" sense.