Statistics 251/551 spring 2013
2013: Solutions to sheet 5

If you are not able to solve a part of a problem, you can still get credit for later parts: Just assume the truth of what you were unable to prove in the earlier part.

[1] *Suppose σ and τ are stopping times for a "flow of information" $W_0, W_1, \dots$. Which of the following are necessarily stopping times? In each case give a proof or counterexample.*

(i) *(5 points)* $\max(\sigma, \tau)$

(ii) *(5 points)* $\min(\sigma, \tau)$

(iii) *(5 points)* $\tau + \sigma$

(iv) *(5 points)* $(\tau - \sigma)^+ := \max(\tau - \sigma, 0)$

First note that if τ is a stopping time then, for each integer k, the indicator

$$\mathbb{I}\{\tau \le k\} = \sum\nolimits_{i \le k} \mathbb{I}\{\tau = i\}$$

is also a function of $W_{0,k}$. Conversely, if $\mathbb{I}\{\tau \le k\} = g_k(W_{0,k})$ for each k then $\mathbb{I}\{\tau = k\} = g_k(W_{0,k}) - g_{k-1}(W_{0,k-1})$, which is also a function of $W_{0,k}$. (For $k = 0$ interpret g_{k-1} as the zero function.) That is, τ is a stopping time if and only if $\mathbb{I}\{\tau \le k\}$ is a function of $W_{0,k}$ for each integer k.

Back to the Problem. For each integer k,

$$\begin{aligned}
\mathbb{I}\{\max(\sigma, \tau) \le k\} &= \mathbb{I}\{\sigma \le k\}\mathbb{I}\{\tau \le k\} \\
\mathbb{I}\{\min(\sigma, \tau) \le k\} &= \max\left(\mathbb{I}\{\sigma \le k\}, \mathbb{I}\{\tau \le k\}\right) \\
\mathbb{I}\{\tau + \sigma = k\} &= \sum\nolimits_{j=0}^{k} \mathbb{I}\{\sigma = j\}\mathbb{I}\{\tau = k - j\}.
\end{aligned}$$

In each case, the expression on the right-hand side only involves $W_{0,k}$.

As a general proposition, $(\tau - \sigma)^+$ need not be a stopping time. For example, suppose $\tau = \min\{n : W_n \ge 1\}$ with W_n as in the next Problem and σ is always equal to 1, a trivial sort of stopping time. Then

$$\mathbb{I}\{(\tau - \sigma)^+ = 1\} = \mathbb{I}\{\tau = 2\} = \mathbb{I}\{W_0 < 1, W_1 < 1, W_2 \ge 1\},$$

which clearly depends on more than W_0 and W_1. There are however some pairs of stopping times σ' and τ' for which $(\tau' - \sigma')^+$ is a stopping time: consider $\tau' = \sigma + \tau$ and $\sigma' = \sigma$, for example.

Some of you argued (incorrectly, or at least incompletely) as follows in part (i) for the time $\gamma := \max(\sigma, \tau)$. "Either $\gamma = \sigma$ so that $\{\gamma = k\} = \{\sigma = k\}$, which only depends on $W_{0,k}$, or $\gamma = \tau$ so that $\{\gamma = k\} = \{\tau = k\}$, which also only depends on $W_{0,k}$." The error lies in the hidden effects of the event $\{\sigma \geq \tau\}$ and its complement. Indicator functions make the difficulty more visible:

$$\mathbb{I}\{\gamma = k\} = \mathbb{I}\{\sigma = k\}\mathbb{I}\{\sigma \geq \tau\}\mathbb{I}\{\tau = k\}\mathbb{I}\{\sigma < \tau\}.$$

One way to complete the argument is to expand the right-hand side as

$$\mathbb{I}\{\sigma = k\} \sum\nolimits_{i \leq k} \mathbb{I}\{\tau = i\} + \mathbb{I}\{\tau = k\} \sum\nolimits_{i < k} \mathbb{I}\{\sigma = i\}$$

then argue that all the indicators can be written as functions of $W_{0,k}$. Effectively you need something like the first paragraph of my solution.

[2] Consider a random walk on the integers $\mathbb{Z}$ defined by $W_n = W_0 + \sum_{1 \leq i \leq n} \xi_i$ where $W_0, \xi_1, \xi_2, \ldots$ are independent random variables with

$$\mathbb{P}\{\xi_i = +1\} = 1/2 = \mathbb{P}\{\xi_i = -1\}.$$

For each m in $\mathbb{Z}$ define

$$\sigma_m = \begin{cases} \min\{n \in \mathbb{N} : W_n = m\} & \\ +\infty & \text{if } W_n \neq m \text{ for all } n \in \mathbb{N} \end{cases}$$

and $\tau := \min(\sigma_0, \sigma_N)$ where $N := a+b$, with a and b positive integers. As in the class presentation of the gambler's ruin problem, write V for the event $\{\tau = \sigma_N\}$, which corresponds to A winning. Write θ_i for $\mathbb{P}(V \mid W_0 = i)$ and λ_i for $\mathbb{E}(\tau \mid W_0 = i)$ if $0 \leq i \leq N$.

This Problem started out as something more complicated (with $p \neq 1/2$), which I decided to simplify. Unfortunately (or fortunately, depending on how you look at it), I forgot about Chang page 126.

(i) (10 points) Show that $W_0, W_1, \ldots$ is a martingale. Use the optional sampling theorem for martingales to show that $\theta_a = a/N$. (Don't forget to start with $\tau \wedge k$ for a positive integer k.)

For each n and $\boldsymbol{i} = (i_0, i_1, \ldots, i_n)$,

$$\mathbb{E}(W_{n+1} \mid W_{0,n} = \boldsymbol{i}) = i_0 + \cdots + i_n + \mathbb{E}(\xi_{n+1} \mid W_{0,n} = \boldsymbol{i})$$

The last term is zero: the conditioning is irrelevant because ξ_{n+1} is independent of $W_0, \xi_1, \dots, \xi_n$ and $\mathbb{E}\xi_{n+1} = 0$.

For each integer k,

$$\begin{aligned} a = \mathbb{E}_a W_0 = \mathbb{E} W_{\tau \wedge k} \\ &= \mathbb{E}_a W_\tau \mathbb{I}\{k \geq \tau = \sigma_0\} + \mathbb{E}_a W_\tau \mathbb{I}\{k \geq \tau = \sigma_N\} + \mathbb{E}_a W_\tau \mathbb{I}\{k < \tau\} \end{aligned}$$

In the last line the first term equals $0 \times \mathbb{P}_a\{k \geq \tau\} V^c = 0$. The second term equals $N \times \mathbb{P}_a\{k \geq \tau\} V$, which tends to $N \mathbb{P}_a V = N\theta_a$ as $k \to \infty$. The third term is bounded (in absolute value) by $N\mathbb{P}\{\kappa < \tau\}$, which tends to zero as $k \to \infty$.

(ii) (10 points) Define $Z_n := W_n^2 - n$. Show that $Z_0, Z_1, \dots$ is a martingale.

For each $n \geq 0$,

$$\begin{aligned} \mathbb{E}(Z_{n+1} \mid W_{0,n}) &= \mathbb{E}\left((W_n + \xi_{n+1})^2 - n - 1 \mid W_{0,n})\right) \\ &= W_n^2 - n + 2W_n \mathbb{E}(\xi_{n+1} \mid W_{0,n}) + \mathbb{E}(\xi_{n+1}^2 \mid W_{0,n}) - 1 \end{aligned}$$

The W_n's are treated like constants when conditioning on $W_{0,n}$. Again the conditioning is irrelevant for the other terms in the last line, which therefore reduces to

$$Z_n + (2W_n \times 0) + 1 - 1 = Z_n.$$

As some of you noted, $\{Z_n\}$ is also a martingale for the flow of information $Z_0, Z_1, \dots$.

(iii) (10 points) Use the optional sampling theorem for martingales to show that $a^2 = N^2\theta_a - \lambda_a$ then deduce that $\lambda_a = ab$. Hint: $Z_{\tau \wedge k} = W_{\tau \wedge k}^2 - \tau \wedge k$.

Start from

$$a^2 = \mathbb{E}_a Z_0 = \mathbb{E}_a Z_{\tau \wedge k} = \mathbb{E}_a W_{\tau \wedge k}^2 - \mathbb{E}_a(\tau \wedge k).$$

It is better to split the $Z_{\tau \wedge k}$ into a difference of two terms before decomposing τ as in part (i). Otherwise you will end up having to explain why $k\mathbb{P}_a\{k < \tau\}$ tends to zero as k tends to infinity.

First note that

$$\mathbb{E}_a(\tau \wedge k) = \mathbb{E}_a W_{\tau \wedge k}^2 - a^2$$

The quantity on the left-hand side increases to λ_a as k goes to infinity. Now decompose τ as in part (i) to rewrite $\mathbb{E}_a W^2_{\tau\wedge k}$ as

$$\mathbb{E}_a 0^2 \mathbb{I}\{k \geq \tau = \sigma_0\} + \mathbb{E}_a N^2 \mathbb{I}\{k \geq \tau = \sigma_N\} + \mathbb{E}_a W_k^2 \mathbb{I}\{k < \tau\}$$

The first term is zero; the second equals $N^2 \mathbb{P}_a V \cap \{\tau \leq k\}$, which tends to $N^2\theta_a$ as $k \to \infty$; the third term is bounded by $N^2 \mathbb{P}_a\{\tau > k\}$, which tends to zero. In the limit we have

$$\lambda_a = N^2\theta_a - a^2 = Na - a^2 = ba.$$

(iv) (10 points) Write δ_i for $\lambda_i - \lambda_{i-1}$. Use Markov chain methods to show that $\delta_{i+1} = \delta_i - 2$ for $i = 1, 2, \ldots, N-1$.

For $0 < i < N$, conditioning on the first step gives

$$\lambda_i = \mathbb{E}_i \tau = 1 + \tfrac{1}{2}\lambda_{i+1} + \tfrac{1}{2}\lambda_{i-1}$$

which rearranges to

$$\tfrac{1}{2}(\lambda_i - \lambda_{i-1}) = 1 + \tfrac{1}{2}(\lambda_{i+1} - \lambda_i).$$

Note also that $\lambda_0 = \lambda_N = 0$.

(v) (10 points) Show that $\lambda_i = \sum_{j=1}^{i} \delta_j = i\delta_1 - (i-1)i$ for $1 \leq i \leq N$. Deduce that $\lambda_i = i(N-i)$ for $0 \leq i \leq N$, in agreement with (iii).

At this point I expected you to solve the difference equations to determine the δ_i's and then recover (by a method different from part (iii)) the solution for the λ_i's. Many of you actually used $\lambda_a = ab$ in order to find the δ_i's, which made the whole exercise rather pointless as an alternative derivation.

Here is what I intended. Repeated substituion gives $\delta_{i+1} = \delta_1 - 2i$ for $1 \leq i \leq N-1$ and

$$\lambda_i - \lambda_0 = \sum\nolimits_{j=1}^{i} \delta_j = i\delta_1 - 2\sum\nolimits_{1\leq j\leq i}(j-1).$$

In particular, for $i = N$ we get

$$0 = \lambda_N - 0 = N\delta_1 - (N-1)N,$$

which implies $\delta_1 = N - 1$ and $\lambda_i = i(N-1) - i(i-1) = i(N-i)$.