Statistics 251/551 spring 2013
2013: Solutions to sheet 6

If you are not able to solve a part of a problem, you can still get credit for later parts: Just assume the truth of what you were unable to prove in the earlier part.

[1] *(15 points) Suppose*

(a) *$S_0, S_1, \dots$ is a supermartingale for a flow of information $W_0, W_1, \dots$*

(b) *σ and τ are stopping times for which $0 \le \sigma \le \tau \le N$, where N is a fixed (finite) positive integer*

(c) *B is a nonnegative random variable for which $B\mathbb{I}\{\sigma = j\}$ depends only on $W_{0,j}$ for $0 \le j \le N$. That is,*

$$B = \sum\nolimits_{0\le j\le N} b_j(W_{0,j})\mathbb{I}\{\sigma = j\}$$

for some nonnegative b_j functions.

Show that $\mathbb{E}S_\sigma B \ge \mathbb{E}S_\tau B$. Explain each step in your reasoning. Hint: Write $\mathbb{E}(S_\tau - S_\sigma)B$ as a double sum over t (for the increments) and j (for the value of σ).

As for the proof of optional stopping, write Δ_t for $S_t - S_{t-1}$. Then

$$\begin{aligned}\mathbb{E}(S_\sigma - S_\tau)B &= \sum\nolimits_{t=1}^{N} \mathbb{E}\Delta_t \left(\mathbb{I}\{t \le \sigma\} - \mathbb{I}\{t \le \tau\}\right) B \\ &= -\sum\nolimits_{t=1}^{N}\sum\nolimits_{j=0}^{N} \mathbb{E}\Delta_t\mathbb{I}\{\tau \ge t > \sigma = j\}b_j(W_{0,j})\end{aligned}$$

Only terms with $j < t$ can give a nonzero value for $\mathbb{I}\{\tau \ge t > \sigma = j\}$. Inside the expectations, Δ_t is always multiplied by the nonnegative function

$$\mathbb{I}\{\tau \ge t > \sigma = j\}b_j(W_{0,j} = \left(1 - \mathbb{I}\{\tau \le t-1\}\right)\mathbb{I}\{t-1 \ge \sigma = j\}b_j(W_{0,j}),$$

a product of three terms that only depend on $W_{0,t-1}$. The supermartingale property shows that the resulting expected values are all ≤ 0.

Many of you failed to treat σ and τ as random, writing things like

$$\mathbb{E}\sum\nolimits_{t=0}^{N-\sigma}(\dots) = \sum\nolimits_{t=0}^{N-\sigma}\mathbb{E}(\dots),$$

which leaves the right-hand side random. You might find it helpful to ponder an analogous mistake in Calculus:

$$\int_0^1 x^2\,dx = x\int_0^1 x\,dx = x/2 \quad ??????????$$

Why can't an x be taken outside the integral?

[2] *(10 points) Near the top of page 2 of the optimal.pdf handout I asserted that the Y process defined at the bottom of page 1 is actually the smallest supermartingale for which $Y_t \geq Z_t$ for each t. Prove that fact.*

Suppose $W_1, W_2, \dots, W_N$ is a supermartingale with $W_t \geq Z_t$ for all t. By construction, $W_N \geq Y_N = Z_N$. Then work backwards inductively: if $W_i \geq Y_i$ for all i such that $t < i \leq N$ then $W_{t+1} \geq Y_{t+1}$ and

$$W_t \geq \mathbb{E}(W_{t+1} \mid Z_{1,t}) \geq \mathbb{E}(Y_{t+1} \mid Z_{1,t}),$$

the first inequality coming from the supermartingale property for W, the second from the inductive assumption that $W_{t+1} \geq Y_{t+1}$. Then use the fact that $W_t \geq Z_t$ to deduce $W_t \geq \max(Z_t, \mathbb{E}(Y_{t+1} \mid Z_{1,t}))$.

Many of you tried to argue by contradiction, starting from an assumption that Y is not the smallest supermartingale for which $Y_t \geq Z_t$ for all t. You then incorrectly claimed that there must exist another supermartingale W for which $Y_t > W_t \geq Z_t$ for all t. This claim is not the negation of the assertion "$W_t \geq Y_t$ for all t if W is a supermartingale with $W_t \geq Z_t$ for all t".

[3] *In class I discussed the simple example where $Z_1, \dots, Z_N$ are independent random variables, each distributed Uniform(0,1), and we seek a stopping time σ for which $\mathbb{E}Z_\sigma$ is maximized. Write M_N for $\max_{i\leq N} Z_i$.*

Oh bother! The symbol M was already being used for the martingale $Y_{\cdot\wedge\tau}$. To avoid further confusion I'll write M_N^* for $\max_{i\leq N} Z_i$.

(i) (10 points) Express $\mathbb{E}M_N^*$ as a function of N.

$$\mathbb{E}M_n = \int_0^1 \mathbb{P}\{\max_{i\leq N} Z_i > y\}\,dy = \int_0^1 (1-y^n)\,dy = \frac{N}{N+1}$$

It was also OK to just remember that M_n has a beta distribution (but we then expected you to give the parameters for that beta) and then quote the expected value.

(ii) (10 points) For $N = 5$ calculate the constants C_i for which $Y_i = \max(Z_i, C_i)$ and

$$\tau := \min\{i \leq N : Z_i = Y_i\}$$

defines the optimal stopping time. (Please explain your reasoning and display your results in a form that is easy for us to read.)

First check that if $Z \sim \text{Uniform}(0,1)$ and c is a constant with $0 \le c \le 1$ then

$$\mathbb{E}\max(Z,c) = \int_0^c c\,dy + \int_c^1 y\,dy = (1+c^2)/2 =: g_0(c)$$

and

$$\mathbb{E}\left(Z\mathbb{I}\{Z \ge c\}\right) = \int_c^1 y\,dy = (1-c^2)/2 =: g_1(c) = 1 - g_0(c)$$

By definition $Y_N = Z_N = \max(Z_N, C_N)$ where $C_N = 0$. By the independence of the Z_i's,

$$\mathbb{E}(Y_N \mid Z_{1,N-1}) = \mathbb{E}Z_N = g_0(C_N) = 1/2.$$

Thus $Y_{N-1} = \max\left(Z_{N-1}, C_{N-1}\right)$ where $C_{N-1} = g_0(C_N)$. Similarly

$$\mathbb{E}(Y_{N-1} \mid Z_{1,N-2}) = \mathbb{E}\max\left(Z_{N-1}, C_{N-1}\right) = g_0(C_{N-1})$$

so that $Y_{N-2} = \max(Z_{N-2}, C_{N-2})$ where $C_{N-2} := g_0(C_{N-1})$. And so on.

More formally, a backwards induction shows that

$$Y_i = \max(Z_i, C_i) \qquad \text{where } C_i := \mathbb{E}\left(Z_{i+1} \vee C_{i+1}\right) = g_0(C_{i+1})$$

That is the constants are defined recursively by $C_N = 0$ and $C_i = g_0(C_{i+1})$ for $1 \le i < N$. As calculated using R:

C1	C2	C3	C4	C5
0.742	0.695	0.625	0.500	0

Notice that $Z_i = Y_i$ iff $Z_i \ge C_i$ so that $\tau = \min\{i : Z_i \ge C_i\}$.

(iii) (10 points) Calculate $\mathbb{E}Z_\tau$. Again, explain your reasoning; a single number will not suffice.

You could start from $\mathbb{E}Z_\tau = \sum_{i=1}^N \mathbb{E}Z_i\mathbb{I}\{\tau = i\}$. By independence of the Z_j's, the ith term equals

$$\mathbb{E}Z_i\mathbb{I}\{Z_1 < C_1, Z_2 < C_2, \ldots, Z_i \ge C_i\} = C_1C_2\ldots C_{i-1}g_1(C_i)$$

That is,

$$\mathbb{E}Z_\tau = g_1(C_1) + C_1g_1(C_2) + + \cdots + C_1C_2\ldots C_{N-1}g_1(C_N)$$

Some of you seemed to be assuming that the event $\{\tau = i\}$ is independent of the random variable Z_i, which is false. Some of you confused $\mathbb{E}(Z_i \mid \tau = i)$ with $\mathbb{E}\left(Z_i \mathbb{I}\{\tau = i\}\right)$.

Here are the values of $\mathbb{E}Z_\tau$ (opt) for various values of N. The first line (max) shows the corresponding values for $\mathbb{E}M_N^*$.

	N=2	N=3	N=4	N=5	N=6	N=7	N=8	N=9	N=10
max	0.667	0.750	0.800	0.833	0.857	0.875	0.889	0.900	0.909
opt	0.625	0.695	0.742	0.775	0.800	0.820	0.836	0.850	0.861

Some of you used a cleverer method. From the handout,

$$\mathbb{E}Z_\tau = \mathbb{E}Y_\tau = \mathbb{E}M_\tau = \mathbb{E}M_1,$$

the last equality coming from the martingale property for M. By construction and the definition of g_0,

$$\mathbb{E}M_1 = \mathbb{E}Y_{1\wedge\tau} = \mathbb{E}Y_1 = \mathbb{E}\max(Z_1, C_1) = g_0(C_1)$$

Presumably some simple algebra would show why the two expressions for $\mathbb{E}Z_\tau$ are equal.

(iv) (5 points) Explain why your answer from (i) for $N = 5$ is bigger than your answer from (ii).

Of course $Z_\tau \le M_N^*$, which implies $\mathbb{E}Z_\tau \le \mathbb{E}M_N^*$. The strict inequality reflects the fact that we can't know whether $Z_\tau = M_N^*$ at time τ if $\tau < N$. Put another way, if $\tau = i < N$ because $Z_i \ge C_i$ there is still a positive probability that $\max_{j:j>i} Z_j$ is larger than Z_i, even though the expected values suggest that there is no point in continuing beyond τ. For example, if $Z_1 = 0.9$ then $\tau = 1$ but there is a positive probability that $Z_2 > 0.9$.