Statistics 330/600b **Due: Thursday 20 January**
2000: Sheet 1

Please attempt at least the starred problems.

*(1.1) Show that the sigma-field on $\mathbb{R}^2$ generated by the class $\mathcal{E}$ of all closed rectangles (with sides parallel to the coordinate axes) contains every closed subset of $\mathbb{R}^2$. Hint: Consider an infinite sheet of graph paper. Shade in those squares that have nonempty intersection with the closed set. Then buy better graph paper.

(1.2) Let $f_1, \ldots, f_n$ be functions in $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$, and let μ be a measure on $\mathcal{A}$. Show that

$$\mu \bigvee_i f_i \le \sum_i \mu f_i \le \mu \bigvee_i f_i + \sum_{i \ne j} \mu(f_i \wedge f_j)$$

where $\bigvee$ denotes pointwise maxima of functions and $\wedge$ denotes pointwise minima.

*(1.3) Suppose T is a function from a set $\mathcal{X}$ into a set $\mathcal{Y}$, and suppose that $\mathcal{Y}$ is equipped with a σ-field $\mathcal{B}$. Define $\mathcal{A}$ as the class of sets of the form $T^{-1}B$, with B in $\mathcal{B}$. Suppose $f \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$. Show that there exists a $\mathcal{B}\backslash\mathcal{B}[0,\infty]$-measurable function g from $\mathcal{Y}$ into $[-\infty, \infty]$ such that $f(x) = g(T(x))$, for all x in $\mathcal{X}$, by following these steps.

(i) Show that $\mathcal{A}$ is a σ-field on $\mathcal{X}$. *It is called the σ-field generated by the map T. It is often denoted by $\sigma(T)$.*

(ii) Show that $\{f \ge i/2^n\} = T^{-1}B_{i,n}$ for some $B_{i,n}$ in $\mathcal{B}$. Define

$$f_n = 2^{-n} \sum_{i=1}^{4^n} \{f \ge i/2^n\} \qquad \text{and} \qquad g_n = 2^{-n} \sum_{i=1}^{4^n} B_{i,n}.$$

Show that $f_n(x) = g_n(T(x))$ for all x.

(iii) Define $g(y) = \limsup g_n(y)$ for each y in $\mathcal{Y}$. Show that g has the desired property. Question: Why can't we define $g(y) = \lim g_n(y)$?