Statistics 330/600b **Due: soon**
2000: Sheet 10

A student wishing to improve his or her grade for the course should attempt the problems on this sheet. Of course, no change is promised, but it is guaranteed that the final grade will be no lower than the tentative grade. Please email DP if you intend to submit solutions to some or all of the problems on this sheet.

(10.1) Let P_0 and Q_0 be probability measures on the real line, for which the functions $M(\theta) := P_0^x \exp(\theta x)$ and $N(\theta) := Q_0^y \exp(\theta y)$ are finite for all θ in the interval $(-\delta, \delta)$. Define P_θ, for $|\theta| < \delta$, to be the probability measure with density $\exp(\theta x)/M(\theta)$ with respect to P_0. Define Q_θ similarly, via the density $\exp(\theta x)/N(\theta)$ with respect to Q_0. Show that the convolution $P_\theta \star Q_\theta$ is absolutely continuous with respect to the convolution $P_0 \star Q_0$, and find its density. Hint: Reread Section 4.4.

(10.2) Let $\mathcal{C}_0(\mathbb{R}^k)$ denote the set of all continuous functions on $\mathbb{R}^k$ with compact support (that is, a continuous function f belongs to $\mathcal{C}_0(\mathbb{R}^k)$ if $\{f \neq 0\}$ is a bounded set). Suppose P and Q are probability measures for which $Pf = Qf$ for all f in $\mathcal{C}_0(\mathbb{R}^k)$. Show that $P = Q$, as measures on $\mathcal{B}(\mathbb{R}^k)$.

(10.3) Say that a real valued random variable X is degenerate if there exists some finite constant C for which $X = C$ almost surely.

(i) Let X be a (real valued) random variable for which, for each bounded subinterval J of the real line, $\mathbb{P}\{X \in J\}$ is either 0 or 1. Show that X is degenerate.

(ii) Suppose X and Y are independent (real valued) random variables for which $X(\omega) + Y(\omega) = 1$ for every ω in the underlying probability space. Show that both X and Y are degenerate.

(iii) Repeat part (ii) under the weaker assumption that the independent random variables satisfy the equality $X + Y = 1$ only on a set with probability one.

(10.4) Let P and P_n, for $n = 1, 2, \ldots$, be probability measures on $\mathcal{B}(\mathbb{R}^k)$ for which $P_n \rightsquigarrow P$ and $P_n^x |x|^2 \to P^x |x|^2 < \infty$. Show that $P_n f \to Pf$ for every continuous real function f on $\mathbb{R}^k$ for which $f(x) = O(|x|^2)$ as $|x| \to \infty$. Hint: Consider $P\left(|x|^2 - M\right)^+$ for large values of M. Also note that $T \wedge M + (T - M)^+ = T$ for each real M and T.

(10.5) Let $0 = X_0, X_1, \ldots$ be nonnegative integrable random variables that are adapted to a filtration $\{\mathcal{F}_i\}$. Suppose there exist constants θ_i, with $0 \le \theta_i \le 1$, for which $\mathbb{P}(X_i \mid \mathcal{F}_{i-1}) \ge \theta_i X_{i-1}$ for $i \ge 1$, that is, there exist nonnegative, $\mathcal{F}_{i-1}$-measurable random variables Y_{i-1} for which $\mathbb{P}(X_i \mid \mathcal{F}_{i-1}) = Y_{i-1} + \theta_i X_{i-1}$ almost surely. Let $C_1 \ge C_2 \ge \ldots \ge C_{N+1} = 0$ be constants. Prove the inequality

$$(**) \qquad \mathbb{P}\{\max_{i \le N} C_i X_i \ge 1\} \le \sum_{i=1}^{N} (C_i - \theta_{i+1} C_{i+1}) \mathbb{P} X_i,$$

by following these steps.

(i) Put $Z_i = X_i - Y_{i-1} - \theta_i X_{i-1}$. Show that $C_i X_i \le C_{i-1} X_{i-1} + C_i Z_i + C_i Y_{i-1}$ almost surely.

(ii) Deduce that $C_i X_i \le M_i + \Delta$, where M_i is a martingale with $M_0 = 0$ and $\Delta = \sum_{i=1}^{N} C_i Y_{i-1}$.

(iii) Show that the left-hand side of inequality $(**)$ is less than $\mathbb{P} C_\tau X_\tau$ for an appropriate stopping time τ, then rearrange the sum for $\mathbb{P}\Delta$ to get the asserted upper bound.