Statistics 330/600b **Due: Thursday 3 February**
2000: Sheet 3

Please attempt at least the starred problems.

*(3.1) Let $\{h_n\}$, $\{f_n\}$, and $\{g_n\}$ be sequences of μ-integrable functions that converge μ almost everywhere to limits h, f and g. Suppose $h_n(x) \le f_n(x) \le g_n(x)$ for all x. Suppose also that $\mu h_n \to \mu h$ and $\mu g_n \to \mu g$. Adapt the proof of Dominated Convergence to prove that $\mu f_n \to \mu f$.

*(3.2) Let μ be a finite measure on the Borel sigma-field $\mathcal{B}(\mathcal{X})$ of a metric space $\mathcal{X}$. Call a set B ***inner regular*** if $\mu B = \sup\{\mu F : B \supseteq F \text{ closed }\}$ and ***outer regular*** if $\mu B = \inf\{\mu F : B \subseteq G \text{ open }\}$

(i) Prove that the class $\mathcal{B}_0$ of all Borel sets B for which both B and B^c are inner regular is a sigma-field. Deduce that every Borel set is inner regular.

(ii) Suppose μ is tight: for each $\epsilon > 0$ there exists a compact K_ϵ such that $\mu K_\epsilon^c < \epsilon$. Show that the F in the definition of inner regularity can then be assumed compact.

(3.3) Suppose a class of sets $\mathcal{E}$ cannot separate a particular pair of points x, y: for every E in $\mathcal{E}$, either $\{x, y\} \subseteq E$ or $\{x, y\} \subseteq E^c$. Show that $\sigma(\mathcal{E})$ also cannot separate the pair.

(3.4) Let $A_1, A_2, \ldots$ be events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define $X_n = A_1 + \ldots + A_n$ and $\sigma_n = \mathbb{P}X_n$. Suppose $\sigma_n \to \infty$ and $\mathbb{P}X_n^2/\sigma_n^2 \to 1$. *(Compare with the inequlity $\mathbb{P}X_n^2 \ge \sigma_n^2$, which follows from Jensen's inequality.)*

(i) Show that

$$\{X_n = 0\} \le \frac{(k - X_n)(k + 1 - X_n)}{k(k+1)}$$

for each positive integer k.

(ii) By an appropriate choice of k (depending on n) in (i), and a passage to the limit, deduce that $\sum_1^\infty A_i \ge 1$ almost surely. Hint: What is the limit of $\{X_n = 0\}$ as n tends to infinity?

(iii) Prove that $\sum_m^\infty A_i \ge 1$ almost surely, for each fixed m. Hint: Show that the two convergence assumptions also hold for the sequence $A_m, A_{m+1}, \ldots$.

(iv) Deduce that $\mathbb{P}\{\omega \in A_i \text{ for infinitely many } i\} = 1$.

(v) If $\{B_i\}$ is a sequence of events for which $\sum_i \mathbb{P}B_i = \infty$ and $\mathbb{P}B_iB_j = \mathbb{P}B_i\mathbb{P}B_j$ for $i \ne j$, show that $\mathbb{P}\{\omega \in B_i \text{ for infinitely many } i\} = 1$.